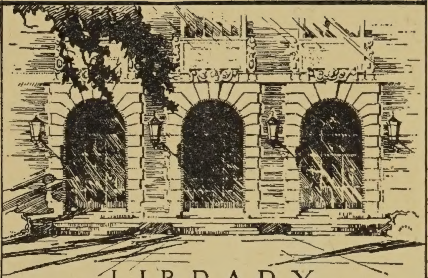

This method of expanding a determinant is evidently general. From it we can easily see that if ~~if one~~ only one element of Δ ~~of~~ of the n^{th} order is zero Δ has $n!-(n-1)!$ significant terms. If ~~two or~~ only two elements vanish, the number of significant terms is either $n!-2(n-1)!$ or $n!+(n-2)!-2(n-1)!$ If only three vanish $(n\geq 3)$ this number is represented by one of the following formulas

$n!-3(n-1)!$

$n!+2(n-2)!-3(n-1)!$

$\cancel{n!+(n-3)!+3(n-2)!-3(n-1)!}$

$n!+3(n-2)!-(n-3)!-3(n-1)!$

II. when ~~a~~ numerical Δ of the 4th or even a higher order involves many zeros it may sometimes be very easily expended directly without reducing it to a lower order.

~~This is The value of the numerator of eq 7 the value of x in Eq 9 may be reduced obtained directly as follows:~~

The following determinant may be used as an illustration.

$$\begin{vmatrix} 0 & 17 & 1 & -9 \\ 0 & 0 & 3 & 2 \\ 1 & -8 & 0 & 7 \\ -3 & 5 & 0 & 0 \end{vmatrix}$$

We shall develop ~~this~~ Δ by ~~observing~~ selecting all the finite terms which

have the elements of the
second column we see
directly that there is

G. A. Miller

only one finite term
which contains 17; viz

$-(-3\times 17\times 3\times 7)= \cancel{1063}\ 1071$

The sign of this term is
negative because the subscripts
of if expressed ~~would~~ would
be 4, 1, 2, 3,

The two finite terms containing
-8 are

$+(-3\times -8\times 3\times -9)=-648$

$-(-3\times -8\times 1\times 2)= -48$

The two finite terms containing 5 are

$-(1\times 5\times 3\times -9)= 135$

$1\times 5\times 1\times 2 = 10$

Hence $\Delta = 520$.

IV.

The number of inversions in the secondary diagonal if the letters are kept in their natural order is $\frac{n(n-1)}{2}$, this is even when n or $n-1$ is divisible by 4.

DETERMINANTS.

AN INTRODUCTION TO THE STUDY OF,
WITH EXAMPLES AND
APPLICATIONS.

BY

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1892.

v1

Show that

$$\begin{vmatrix} 2 & 1 & 3 & 4 \\ 3 & 2 & 1 & 2 \\ 1 & 0 & 5 & 6 \\ 4 & -1 & 3 & 2 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 1 & -5 & 2 \\ -2 & 5 & -2 \\ -6 & 6 & -6 \end{vmatrix} =$$

$$\cancel{-2 \begin{vmatrix} 1 & -5 & 2 \\ -2 & & \end{vmatrix}} \quad -2\begin{vmatrix} 1 & -5 & 2 \\ -1 & 0 & 0 \\ 4 & -1 & 1 \end{vmatrix} = \frac{2}{5}\begin{vmatrix} -5 & 0 \\ 4 & -3 \end{vmatrix}$$

CONTENTS.

DETERMINANTS.

History. — The earliest known record of determinants is found in the writings of Leibnitz, who lived 1646–1716. He communicated his discoveries to L'Hospital in a letter dated April 28, 1693, and in a later letter he expressed his belief that their study would lead to many valuable discoveries. We have no evidence that he or his correspondent pursued the study beyond a few of the most elementary principles. No trace of determinants can be found in the writings of the mathematicians who succeeded Leibnitz until 1750. In this year Cramer rediscovered them while working on the analysis of curves. From 1750 to 1826 the subject was studied by only a few eminent mathematicians, whose writings were suited only for advanced students. Among these writers were Bézout, Laplace, Lagrange, Gauss, Cauchy, etc.

In 1826 Jacobi commenced a series of papers on this subject in "Crelle's Journal." These papers continued for nearly twenty years, and by them the subject was made available for ordinary students. Many new and important theorems were also added. In late years the subject has been enriched by the writings of many mathematicians, pre-eminent among whom are Sylvester and Cayley.

Nature. — Given the equations

$$\left.\begin{aligned} a_1x + b_1y &= m_1 \\ a_2x + b_2y &= m_2 \end{aligned}\right\} \text{A.}$$

Multiplying the first of these equations by b_2, and the second by b_1, we have

$$\left.\begin{aligned} a_1b_2x + b_1b_2y &= b_2m_1 \\ a_2b_1x + b_1b_2y &= b_1m_2 \end{aligned}\right\} \text{A}'.$$

Subtracting the second equation from the first, the result will be

$$(a_1b_2 - a_2b_1)\, x = b_2m_1 - b_1m_2. \quad \text{B.}$$

By multiplying the first of equations A by a_2, and the second by a_1, and subtracting the first of the resulting equations from the second, we obtain

$$(a_1b_2 - a_2b_1)\, y = a_1m_2 - a_2m_1. \quad \text{C.}$$

From equations B and C the values of x and y are found to be

$$x = \frac{b_2 m_1 - b_1 m_2}{a_1 b_2 - a_2 b_1} \ : \ y = \frac{a_1 m_2 - a_2 m_1}{a_1 b_2 - a_2 b_1}.$$

If the notation $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$ be employed to denote $2 \times 4 - 3 \times 1$; i.e., the difference of the products obtained by multiplying diagonally, we may write

$$x = \frac{\begin{vmatrix} m_1 & b_1 \\ m_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} : \quad y = \frac{\begin{vmatrix} a_1 & m_1 \\ a_2 & m_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}, \quad \begin{vmatrix} m_1 & b_1 \\ m_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

etc., are *determinants.*

Observations. — The denominator is the same for both unknowns, and is a determinant formed by writing the coefficients of the unknowns in order. This determinant is called the *determinant of the system.*

The numerator of the value of x is obtained from the determinant of the system by substituting the second members of the equations (A) in order for the coefficients of x.

The numerator of the value of y is obtained by substituting the same quantities for the coefficients of y.

The product of the elements in the diagonal ending in the upper left-hand corner is positive. This diagonal is called the *principal diagonal.* The product of the elements in the diagonal ending in the upper right-hand corner is negative. This diagonal is called the *secondary diagonal.*

Direction. — Employ the determinant notation in the solution of the following problems.

$$\begin{cases} 4x + 2y = 16. \\ x + 3y = 14. \end{cases}$$

Solution: —

$$x = \frac{\begin{vmatrix} 16 & 2 \\ 14 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} 4 & 16 \\ 1 & 14 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}}.$$

$4 \times 3 - 1 \times 2 = 10$, the denominator.
$16 \times 3 - 14 \times 2 = 20$, the numerator of the value of x.
$4 \times 14 - 1 \times 16 = 40$, the numerator of the value of y. Hence
$x = 20 \div 10 = 2$, $y = 40 \div 10 = 4$.

2 $\begin{cases} 3x + 2y = 7. \\ 4y - x = 7. \end{cases}$ **3** $\begin{cases} 2x + 5y = -5. \\ 7x = -21. \end{cases}$

Suggestions. — Write the equations in the form of equations A before writing the determinants. The direct solution of problem 3 is shorter than this general solution. It has been given to explain a method which will be frequently employed in equations of more than two unknown quantities. Examples 2 and 3 lead to the following determinants: —

$$y = \frac{\begin{vmatrix} 3 & 7 \\ -1 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix}}, \qquad x = \frac{\begin{vmatrix} 7 & 2 \\ 7 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix}};$$

$$y = \frac{\begin{vmatrix} 2 & -5 \\ 7 & -21 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 7 & 0 \end{vmatrix}}, \qquad x = \frac{\begin{vmatrix} -5 & 5 \\ -21 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 7 & 0 \end{vmatrix}}.$$

4 $\begin{cases} 3x + y = 9. \\ 2x + 3y = 13. \end{cases}$ **6** $\begin{cases} 2x + 6y = -1\frac{1}{2}. \\ 4x - 9y = 4. \end{cases}$

5 $\begin{cases} x - 2y = 2. \\ 3x + y = 13. \end{cases}$ **7** $\begin{cases} x + 5y = 20. \\ 3x + 4y = 15. \end{cases}$

Remark. — It is generally unnecessary to write the values of the unknowns in the way indicated, since the multiplication may be performed without changing the equa-

tions when they are written in the ordinary form; e.g., in problem 4 we can easily see that the value of the denominator is 7, $(3 \times 3 - 2 \times 1)$, the numerator of the value of x is 14, $(9 \times 3 - 13 \times 1)$, and the numerator of the value of y is 21, $(3 \times 13 - 2 \times 9)$. The work should generally be performed mentally.

Direction. — Solve the first six of the following problems mentally, and verify the results:

8. $\begin{cases} 3x - 4y = 8. \\ x + 3y = 7. \end{cases}$

9. $\begin{cases} 2x + 3y = 2. \\ 5x - 2y = 1\frac{5}{6}. \end{cases}$

10. $\begin{cases} 4x + y = 5. \\ 2x + 7y = 9. \end{cases}$

11. $\begin{cases} ax + 2y = 10. \\ bx + 3y = 4. \end{cases}$

12. $\begin{cases} ax + by = c. \\ dx + ey = f. \end{cases}$

13. $\begin{cases} mx + y = l. \\ x + ny = k. \end{cases}$

14. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = c. \\ \dfrac{x}{d} + \dfrac{y}{e} = f. \end{cases}$

Suggestion. — The denominator is:

$$\begin{vmatrix} \dfrac{1}{a} & \dfrac{1}{b} \\ \dfrac{1}{d} & \dfrac{1}{e} \end{vmatrix}$$

15. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c. \\ \dfrac{d}{x} - \dfrac{e}{y} = f. \end{cases}$

Suggestion. — Find the values of $\dfrac{1}{x}$ and $\dfrac{1}{y}$; then

invert; the results will be the values of x and y.

16. $$\begin{cases} x - \dfrac{3y - 2 + x}{11} = 1 + \dfrac{45x + 4y}{99} \\ \dfrac{3x + 2y}{6} - \dfrac{y - 5}{4} = \dfrac{11x + 152}{12} \\ \qquad - \dfrac{3y + 1}{2}. \end{cases}$$

Suggestion.—Combine the coefficients of x and y.

17. $$\begin{cases} bcx = cy - 2b. \\ b^2y + \dfrac{a(c^3 - b^3)}{bc} = \dfrac{2b^3}{c} + c^3x. \end{cases}$$

18. $$\begin{cases} \frac{1}{4}x + \frac{1}{3}y = 12. \\ \frac{1}{2}x - \frac{1}{5}y = 2\frac{1}{4}. \end{cases}$$

19. $$\begin{cases} .01x + .05y = 4. \\ .2x + y = 20. \end{cases}$$

Given $$\left.\begin{aligned} a_1x + b_1y + c_1z &= m_1 \\ a_2x + b_2y + c_2z &= m_2 \\ a_3x + b_3y + c_3z &= m_3 \end{aligned}\right\} M.$$

Multiply these three equations respectively by the factors:

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

we obtain

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} x + b_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} y + c_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} z = \quad m_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

$$-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} x - b_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} y - c_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} z = - m_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}.$$

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} x + b_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} y + c_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} z = \quad m_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Add these three equations together. The coefficients of y and z reduce to zeros. Hence the value of

$$x = \frac{m_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - m_2\begin{vmatrix} b_1 & b_1 \\ b_3 & c_3 \end{vmatrix} + m_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}.$$

We shall show by expanding, that the coefficient of y is zero. That the coefficient of z is zero is seen in a similar way.

$$b_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_1b_2c_3 - b_1b_3c_2.$$

$$-b_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = -b_1b_2c_3 + b_2b_3c_1.$$

$$b_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = b_1b_3c_2 - b_2b_3c_1.$$

Since the sum of the second members is zero, the sum of the first members must also be zero.

It should be observed that the multiplier of an equation is the determinant formed by writing the coefficients of y and z in the other equations in order, e.g.:

Let us eliminate y and z from the following equations by the method just employed.

$$\begin{cases} 3x - 2y + z = -2. \\ x + 3y - z = 10. \\ 2x - y + 5z = 13. \end{cases}$$

The multipliers are $\begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}, -\begin{vmatrix} -2 & 1 \\ -1 & 5 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix}.$

The values of the determinants are 14, 9, -1. Multiplying we have:

$$\begin{array}{rcrcrcr} 42x & - & 28y & + & 14z & = & -28. \\ 9x & + & 27y & - & 9z & = & 90. \\ -2x & + & y & - & 5z & = & -13. \end{array}$$

Adding the three equations we have:

$$49x = 49.$$

Direction. — Eliminate y and z from the following equations, then find the value of x.

1. $$\begin{cases} 3x + 2y - z = 27. \\ x - y + 3z = 9. \\ x - 2y - 4z = -36. \\ \qquad\qquad \text{Result, } x = 4. \end{cases}$$

2. $$\begin{cases} x - y + 2z = 8 - a. \\ 2x + y - 3z = a - 5. \\ x + \dfrac{y}{a} + z = 6. \\ \qquad\qquad \text{Result, } x = 2. \end{cases}$$

3. $$\begin{cases} x + 2y = 40. \\ 2x + z = 45. \\ 3y - z = 25. \\ \qquad\qquad \text{Result, } x = 20. \end{cases}$$

Suggestion. — The multipliers are:

$$\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}.$$

When the three equations are written in the ordinary way, the operations should performed mentally, e.g.:

Let the value of x in the following equations be required :

$$\begin{cases} 2x + y - 3\overset{z}{\cancel{x}} = 22. \\ x - 2y - z = -1. \\ 3x + y - 2z = 33. \end{cases}$$

The required multipliers are readily found to be $5, -1, -7$. Multiplying the coefficients of x in order, by these factors, and adding the products, we obtain $-12x$. Treating the absolute terms similarly, we obtain -120. The value of x must be $-120 \div -12 = 10$.

Solve the following equations for x, in this way, and verify the results :

$$\begin{cases} 3x + 2y - z = 6. \\ 2x + 3y - 3z = 1. \\ x - y + 4z = 9. \end{cases}$$

$$\begin{cases} x + y + z = 15. \\ 4x - y - 2z = 10. \\ 2x + 3y - z = 19. \end{cases}$$

Resuming equations M (page 7) : —

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= m_1 \\ a_2x + b_2y + c_2z &= m_2 \\ a_3x + b_3y + c_3z &= m_3 \end{aligned}\right\} \text{M.}$$

Multiplying the three equations respectively by the factors

$$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad -\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

we obtain

$$a_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} x + b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} y + c_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} z = m_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$-a_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} x - b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} y - c_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} z = -m_1 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

$$a_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} x + b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} y + c_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} z = m_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Add these three equations together. The coefficients of x and z reduce to zeroes. Hence

$$y = \frac{m_1 \begin{vmatrix} a_2 & c^2 \\ a_3 & c_3 \end{vmatrix} - m_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + m_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}$$

This result could have been obtained directly from the value of x on page 8 by

interchanging the coefficients of x and y. This is evident, since the value of an unknown in a group of equations depends, not on its name, but on its coefficients; e.g., the value of x in equations M is the same as the value of y in equations N, or the value of z in equations P.

$$\left.\begin{array}{l} a_1x + b_1y + c_1z = m_1 \\ a_2x + b_2y + c_2z = m_2 \\ a_3x + b_3y + c_3z = m_3 \end{array}\right\} \text{M.}$$

$$\left.\begin{array}{l} b_1x + a_1y + c_1z = m_1 \\ b_2x + a_2y + c_2z = m_2 \\ b_3x + a_3y + c_3z = m_3 \end{array}\right\} \text{N.}$$

$$\left.\begin{array}{l} c_1x + b_1y + a_1z = m_1 \\ c_2x + b_2y + a_2z = m_2 \\ c_3x + b_3y + a_3z = m_3 \end{array}\right\} \text{P.}$$

Observe, too, that the value of an unknown in a group of equations is not affected by interchanging all of the coefficients of any or all other unknowns.

We have already seen (pages 7 and 8) how the value of x may be found when three equations involving three unknowns

are given. Finding the value of x in this manner from equations N, we obtain the value of y in M (page 13), and from x in equations P, we find that in M

$$z = \frac{m_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} - m_2 \begin{vmatrix} b_1 & a_1 \\ b_3 & a_3 \end{vmatrix} + m_3 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}}{c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} - c_2 \begin{vmatrix} b_1 & a_1 \\ b_3 & a_3 \end{vmatrix} + c_3 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}}.$$

The forms of the values of the three unknowns in equations M lead to the following practical principle, by which simple, simultaneous equations involving three unknowns may be solved.

The value of any unknown is a fraction whose numerator is obtained by multiplying each absolute term (changing the signs of alternate terms) by the determinant formed by writing the coefficients of the other unknowns in the remaining equations in order.

The denominator is obtained by multiplying the coefficients of the required unknown (changing the signs of alternate terms) by the same determinants.

The form of the value of z obtained in

this way will differ from that just given in having a and b interchanged. Such interchanges are allowable (page 13).

The learner should make himself very familiar with the above rule, since it applies to all simultaneous equations, as will be proved later.

Direction. — Find the value of the unknown underscored in the following equations.

$$\mathbf{1.}\begin{cases} 2x - y - 6\underline{z} = 0. \\ x + 8y - \underline{z} = 9\tfrac{1}{2}. \\ 3x + y + 2\underline{z} = 8. \end{cases}$$

Solution. — The multipliers are

$$\begin{vmatrix} 1 & 8 \\ 3 & 1 \end{vmatrix}, \quad -\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & -1 \\ 1 & 8 \end{vmatrix}.$$

or, -23, -5, 17.

$$z = \frac{-0 \times 23 - 9\tfrac{1}{2} \times 5 + 8 \times 17}{6 \times 23 + 1 \times 5 + 2 \times 17} = \tfrac{1}{2}.$$

Since only the ratio of the multipliers is essential, when they have a common factor it should be rejected; and when some involve fractions all should be multiplied by the L. C. D. of the fractions.

2. $\begin{cases} 3x - 2y + 4z = 15. \\ x + y - z = 3. \\ -2x + 3y + 5z = 18. \end{cases}$

Result, $z = 3$.

3. $\begin{cases} 4x + 9y - 2z = 15. \\ 7x + 3y + 8z = 71. \\ x + 15y - 5z = -6. \end{cases}$

Result, $y = 1$.

4. $\begin{cases} x + 3y - z = 30. \\ 2y + z = 12. \\ x - \frac{3}{5}z = 4. \end{cases}$

Result, $x = 10$.

5. $\begin{cases} 5x + y - 2z = 0. \\ \frac{3}{4}z + x = 7\frac{1}{2}. \\ z + y - 3x = 14. \end{cases}$

Result, $y = \cancel{4}.$ 9 7/9

We shall once more resume equations M.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= m_1 \\ a_2x + b_2y + c_2z &= m_2 \\ a_3x + b_3y + c_3z &= m_3 \end{aligned}\right\} \text{M.}$$

The value of x obtained from these equations is (page 8)

$$x = \frac{m_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - m_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + m_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}$$

The denominator, written in the ordinary way, is

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.$$

The numerator may be obtained from this expression by replacing the a's by corresponding m's. The denominator and numerator are also written as follows : —

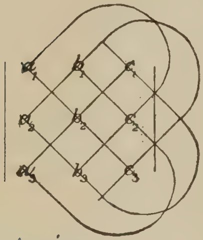
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}.$$

These are determinants of the third order. The determinants previously employed are of the second order. The order depends on the number of elements in one side of the square.

One method of developing a determinant of the third order may be inferred from the expressions from which the above determinants were obtained. The following method is probably more generally used.

Connect the elements of the determinant as indicated in figure (p. 18). Multiply the united elements together. Prefix the positive sign to the products of the elements

in the lines parallel to the principal diagonal, and the negative to those in the lines parallel to the secondary diagonal. The learner may easily verify this rule by comparing the results obtained in this way with the denominator written in the common way.



Definition

Determinants is a method of notation by which large expressions may be written in small forms, exhibiting, in a remarkable manner, the laws which govern the expressions; e.g., the expressions

$a_1b_2 - a_2b_1$; $a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$; and $a_1b_2c_3d_4 - a_2b_1c_3d_4 - a_1b_3c_2d_4 + a_3b_1c_2d_4 + a_2b_3c_1d_4 - a_3b_2c_1d_4 - a_1b_2c_4d_3 + a_2b_1c_4d_3 + a_1b_4c_2d_3 - a_4b_1c_2d_3 - a_2b_4c_1d_3 + a_4b_2c_1d_3 + a_1b_3c_4d_2 - a_3b_1c_4d_2 - a_1b_4c_3d_2 + a_4b_1c_3d_2 + a_3b_4c_1d_2 - a_4b_3c_1d_2 - a_2b_3c_4d_1 + a_3b_2c_4d_1 + a_2b_4c_3d_1 - a_4b_2c_3d_1 - a_3b_4c_2d_1 + a_4b_3c_2d_1$, are written

$$\begin{vmatrix} a_1 & b_3 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

respectively. The larger the expression the more will be gained by writing it in the form of a determinant. While the third of these expressions is so long that it is difficult to get its picture in its entirety well defined in our minds, the equivalent determinant will present no such difficulties. It is this brevity which makes determinants such a strong instrument of investigation.

By the perusal of the preceding introductory pages, the learner should have formed some idea of the nature and uses of determinants, which needs only to be extended. We might have pursued the study in this way, but for simplicity and logical arrangement we shall now change the method, and study the subject from definitions. The preceding pages, while not necessary to the scientific development of the subject, have been given to make the treatment more intelligible, and to

encourage the learner, while investigating many apparently useless principles, by the knowledge that the subject is important. The learner cannot expect to see the value of the subject fully until the elements of it have been so thoroughly mastered that he can make the application readily and intelligently.

The words of Professor Sylvester may aid in forming a correct view of Determinants. He says: "It is an algebra upon an algebra; a calculus which enables us to combine and foretell the results of algebraical operations in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic."

INVERSIONS AND PERMUTATIONS.

1. *Definitions.* — A change from the natural order is called an *inversion.* The different orders in which several things can be put are their *permutations.*

2. When a group of different consecutive integers are written in their natural order, e.g., 1, 2, 3, 4, 5, there are no *inversions.*

3. When these integers are written in any other order, e.g., 2, 1, 3, 5, 4, the number of inversions is determined by the number of times a larger number precedes a smaller.

Illustrations. — In 2, 1, 3, 5, 4 there are two inversions — 2 before 1 and 5 before 4; in 2, 3, 4, 5, 1 there are four inversions — 2 before 1, 3 before 1, 4 before 1, and 5 before 1.

4. Determine the number of inversions in the following groups of integers.

1. 2, 3, 4, 1, 5 : 3, 4, 1, 2, 5 : 1, 4, 5, 3, 2.
Ans. 3, 4, 5.

2. 1, 2, 3, 4 : 2, 3, 4, 1 : 4, 3, 2, 1.
Ans. 0, 3, 6.

3. 1, 2, 3 : 1, 3, 2 : 2, 1, 3 : 2, 3, 1 : 3, 1, 2.
Ans. 0, 1, 1, 2, 2.

4. 1, 2, 3, 4, 5, 6 : 6, 5, 4, 3, 2, 1 : 2, 3, 5, 4, 6, 1.
Ans. 0, 15, 6.

5. If two adjacent integers in the series 1, 2, 3, 4 be interchanged, what is the effect on the number of inversions ?

6. Can the number of inversions be made odd by interchanging two adjacent integers in the group 2, 3, 4, 1, 5 ?

5. The permutations of any given group of numbers are divided into two classes. The first class embraces all the permutations in which there is an even number of inversions, and the second class all those in which there is an odd number of inversions. E.g., 4, 3, 2, 1 : 2, 3, 1 : 3, 1, 2, are permutations of the first class, and 1, 3, 2 : 2, 1, 3 : 2, 3, 4, 1, 5, are permutations of the second class.

6. *Theorem.*—*Any interchange of two numbers in a group of different numbers alters the class of permutation of the group.*

First Part of the Demonstration.

When a number and its neighbor interchange places and all the others remain undisturbed.

In the group 2, 1, 5, . . . 3, 6, 9, . . . 7, 11, interchange any two neighboring numbers, as 6 and 9, and write the two resulting groups as follows :

$$2,\ 1,\ 5,\ \ldots\ 3, \begin{bmatrix} 6, & 9 \\ 9, & 6 \end{bmatrix} \ldots\ 7,\ 11, \text{ and}$$

observe,—

(a) The number of inversions which the

integers outside of the brackets have, with respect to each other, is the same in both groups.

(b) The number of inversions which the integers outside of the brackets (preceding) have, with respect to those within the brackets, is the same.

(c) The number of inversions which the integers within the brackets have with those outside of the brackets (following) is the same.

(d) The number of inversions which the integers within the brackets have, with respect to each other, is changed from 0 to 1, or *vice versa*. Hence, in this case, the class of the permutation is changed.

Second Part of the Demonstration.

When any two numbers in a group interchange places.

In the above group let 5 and 11 interchange places, there being ~~n~~ intermediate $n-1$ numbers. 11 may be brought to the place which 5 now occupies, by n successive left neighbor interchanges. After this is done, 5 may be brought to the place which 11

occupies in the original group by $n-1$ right neighbor interchanges. Each of these neighbor interchanges has altered the number of inversions by unity. The $n+n-1$ interchanges must have altered the number of inversions by an odd number. If the number of inversions in the original permutation was even, it will now be odd; and *vice versa*. Hence, the class of permutation of the group has been altered. Q. E. D.

This important principle is sometimes stated thus: If, in a series of numbers which are all different, any two are interchanged, the others remaining undisturbed, the number of inversions is increased or decreased by an odd number.

Examples.

Interchange the numbers underscored in the following groups, and compare the results with the theorem just proved:

1. $2, \underline{3}, \underline{4}, 1 : 1, 4, \underline{5}, \underline{2}, 3 : \underline{2}, \underline{1} : 3, \underline{1}, \underline{2}.$

2. $1, \underline{4}, 5, \underline{3}, 2 : \underline{2}, 3, \underline{1} : 3, 1, \underline{6}, 4, 5, \underline{2}.$

*7. This theorem is also proved as follows:

* Articles marked with an asterisk may be omitted ~~on a first reading~~ without destroying the rigor or the continuity of the ~~development~~ treatise development of the subject

If all the remainders obtained by subtracting each number from all the preceding numbers be multiplied together, the product will be positive or negative as the permutation of the group is of the first or second class.

Let r and s be any two numbers of the group except the two that are to be interchanged, and i and k be the two to be interchanged. The class of permutation of a given group is determined by

$$(i-k)\,\Pi\,(r-i)\,(r-k)\,\Pi\,(r-s),$$

See proof at end

$\Pi\,(r-s)$ denoting the "product of all such factors." The sign of $\Pi\,(r-i)\,(r-k)$, $\Pi\,(r-s)$ is not affected by interchanging i and k, while the sign of $(i-k)$ will be reversed. The class of permutation of the group will, therefore, be altered by the interchange of i and k. Q. E. D.

8. *Cor.* — If a number is transferred to another place, all the others maintaining their relative positions, the resulting permutation is of the $\left\{\begin{array}{l}\text{the same}\\\text{a different}\end{array}\right.$ class, if the number transferred has moved over an $\left\{\begin{array}{l}\text{odd}\\\text{even}\end{array}\right.$ number of places.

Illustration. — In the permutation 4, 3, 5, 1, 2, transfer 1 to the place of 3; the result will be (4, 1, 3, 5, 2) a permutation of the same class, since 1 has been transferred over an odd number of places (1). Again, transfer 1 to the place of 4; the result will be (1, 4, 3, 5, 2) a permutation of a different class, since 1 has been transferred over an even number of places (~~1~~ 2). 2

This transferrence can be accomplished by successive neighbor interchanges.

Examples.

Move the numbers underscored to the places marked with carets, and determine the resulting permutation by the theorem (neighbor interchanges) and the corollary:

1. $2, {}_{\wedge} 4, 3, \underline{5}, 1$: $\underline{4}, 3, 2, {}_{\wedge} 1$: $2, 4, \underline{6}, 5, 3, 1, {}_{\wedge} 7$.

9. *Theorem.* — The number of permutations of the group 4, 3, 2, . . . 7, 5, composed of n elements is $n!$.

Demonstration. — The first place of this group may be filled in n ways [any one of the given elements may occupy it]. After the first place has been filled, the second

may be filled in $n-1$ ways [any one of the remaining elements may occupy it]. After the first two places have been filled, the third may be filled in $n-2$ ways [any one of the remaining elements may occupy it], etc. Therefore the number of permutations of the group (the different orders in which n thing can be put) is n/

$$n\,(n-1),\ (n-2)\ .\ .\ .\ 2\times 1 = n!.$$

Illustration. — The following are the $4!$ permutations of 1, 2, 3, 4 :

$$\mathbf{1.}\left\{\begin{array}{l} 2\begin{cases}3 & 4\\ 4 & 3\end{cases}\\[1ex] 3\begin{cases}2 & 4\\ 4 & 2\end{cases}\\[1ex] 4\begin{cases}2 & 3\\ 3 & 2\end{cases}\end{array}\right.\qquad \mathbf{2.}\left\{\begin{array}{l} 1\begin{cases}3 & 4\\ 4 & 3\end{cases}\\[1ex] 3\begin{cases}1 & 4\\ 4 & 1\end{cases}\\[1ex] 4\begin{cases}1 & 3\\ 3 & 1\end{cases}\end{array}\right.$$

$$\mathbf{3.}\left\{\begin{array}{l} 1\begin{cases}2 & 4\\ 4 & 2\end{cases}\\[1ex] 2\begin{cases}1 & 4\\ 4 & 1\end{cases}\\[1ex] 4\begin{cases}1 & 2\\ 2 & 1\end{cases}\end{array}\right.\qquad \mathbf{4.}\left\{\begin{array}{l} 1\begin{cases}2 & 3\\ 3 & 2\end{cases}\\[1ex] 2\begin{cases}1 & 3\\ 3 & 1\end{cases}\\[1ex] 3\begin{cases}1 & 2\\ 2 & 1\end{cases}\end{array}\right.$$

Writing them in the common way, we have

1, 2, 3, 4	1, 2, 4, 3	1, 3, 2, 4
2, 1, 3, 4	2, 1, 4, 3	2, 3, 1, 4
3, 1, 2, 4	3, 1, 4, 2	3, 2, 1, 4
4, 1, 2, 3	4, 1, 3, 2	4, 2, 1, 3
1, 3, 4, 2	1, 4, 2, 3	1, 4, 3, 2
2, 3, 4, 1	2, 4, 1, 3	2, 4, 3, 1
3, 2, 4, 1	3, 4, 1, 2	3, 4, 2, 1
4, 2, 3, 1	4, 3, 1, 2	4, 3, 2, 1

10. All the permutations may be formed from any one permutation by the successive interchange of adjacent numbers (see illus.).

Exercise.

1. Verify that the following groups have 2!, 3!, and 5! permutations respectively.

1, 2 : 1, 2, 3 : 1, 2, 3, 4, 5.

2. How many different words of ten letters each could be formed from an alphabet containing ten letters, no letter being used twice in the same word?

3. How many different words, no letter being used twice in the same word?

4. Prove that the following groups have 2!, 3!, and 4![1] combinations respectively, permutations of the letters and of the subscripts being allowed.

$$a_1, b_2 \ : \ a_1, b_2, c_3 \ : \ a_1, b_2, c_3, d_4.$$

11. The class of a permutation is changed by each interchange of two numbers (6). The number of permutation of any group is even, since $n!$ is even when n exceeds unity. Hence the number of permutations in which there is an even number of inversions is equal to the number of permutations in which there is an odd number of inversions (10).

12. This important principle may also be proved as follows: —

A group of n elements has $n!$ permutations. Let x and y represent the number of permutations having an even and an odd number of inversions, respectively. Then $x + y = n!$. In all of the permutations interchange two given elements. All the resulting permutations are different.

[1] The different groups that can be made of n things, without regard to order, are their combinations.

The x permutations have become the y permutations, and *vice versa*, without an increase in terms. $\therefore \quad x = y$. Q. E. D.

* 13. When each element of a given series is replaced by the following element, and the last by the first, the elements are said to be cyclically interchanged. It is evident that a cyclical interchange of a given number of elements (n) may always be effected by $n - 1$ interchanges of two adjacent elements. If the elements

$$a, b, c, d, \quad . \quad . \quad . \quad l$$

were written on the circumference of a circle, and the circumference cut between a and l, the elements would be in their natural order, while the cutting between a and b would cause the elements to be cyclically interchanged. The term cyclical interchange is extended to include the sub. case where each element is replaced by another element of sub- a group, and the last by the first. That is where sub-groups are cyclically interchanged.

* 14. Every permutation of a series may be obtained from a given permutation by cyclical interchanges.

Given	5, 6, 2, 1, 4, 3, 7.
to obtain	2, 3, 7, 6, 4, 5, 1.

Solution.—Replace 5 by 2, 2 by 7, 7 by 1, 1 by 6, 6 by 3, 3 by 5. This constitutes the first cyclical interchange. 4 is said to form a cycle by itself.

It is evident that this method can be employed in all cases.

* 15. Two permutations belong to $\left\{\begin{array}{l}\text{the same class}\\ \text{different classes}\end{array}\right.$ if the difference between the number of elements and the number of groups by whose cyclical interchanges one is obtained from the other is $\left\{\begin{array}{l}\text{even.}\\ \text{odd.}\end{array}\right.$

Demonstration.—Let n be the number of elements, and p the number of cyclical interchanges of $a_1, a_2, a_3, \ldots a_p$ elements respectively. Then must the number of single interchanges be

$(a_1 - 1) + (a_2 - 1) + (a_3 - 1) + \ldots + (a_p - 1) = n - p.$

For $a_1 + a_2 + a_3 + \ldots + a_p = n$ by hypothesis.

In the given example $n = 7$ and $p = 2$,

$\therefore\ n - p =$ an odd number, and the two permutations belong to different classes.

Given	2, 4, 3, 1, 5, 6, 8, 7.
to obtain	1, 2, 4, 3, 6, 8, 7, 5.

Here $n = 8$ and $p = 2$. Hence the two permutations are of the same class.

16. Observe that numerals are not essential to the preceding demonstrations. Any elements — numeral, literal, monomial, polynomial, etc. — may be regarded in their natural order if arranged in one way. The inversions, when they are arranged differently, are found the same way as if they were numerals, and their natural order were the natural order of these numerals; e.g., if $a + b$, d, e, c be considered the natural order, there are four inversions in c, $a + b$, e, d.

Examples of Determinants.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 6 \\ 4 & 3 \end{vmatrix}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

are determinants of the second order.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}, \quad \begin{vmatrix} 2 & 3 & 4 \\ 6 & 0 & 1 \\ 2 & 1 & -5 \end{vmatrix}$$

are determinants of the third order.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 & 4 & -6 \\ 3 & -2 & 6 & 4 \\ 8 & 10 & -4 & 8 \\ 2 & 3 & 5 & -1 \end{vmatrix}$$

are determinants of the fourth order.

$$\begin{vmatrix} a_1 & b_1 & \dots & l_1 \\ a_2 & b_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots \\ a_n & b_n & \dots & l_n \end{vmatrix}$$

is a general determinant; i.e., a determinant of the n^{th} order.

Meaning of this Notation.

I.

18.[1] *Number of Terms.*—The square form of elements between two vertical

[1] These definitions should be regarded arbitrary for the present. The reasons will appear later.

lines represents all the terms which can be formed containing one element and only one of each row (horizontal line) and column (vertical line) as a factor.

II.

19. *Natural Order.* — The elements in the diagonal beginning in the upper left-hand corner are said to be in their natural order. This diagonal is called the Principal Diagonal; the other diagonal is called the Secondary Diagonal.

III.

20. *Subscripts.* — In order that inversions may be readily seen, the subscript will generally indicate the row, and the letter the column, to which an element belongs. This notation does not imply any relation or dependence of the elements. It will, however, prove very helpful in studying the laws governing determinants, and will generally be employed.

IV.

21. *Inversions.* — The number of inversions in a term is determined by the number of inversions in the $\begin{cases}\text{subscripts,}\\ \text{letters,}\end{cases}$ when the $\begin{cases}\text{letters}\\ \text{subscripts}\end{cases}$ are in their natural order, or by the number of inversions in both when neither are in the natural order.

V.

22. *Signs.* — Terms in which the number of inversions is even are positive, the others are negative.

VI.

23. *Arrangement.* — Since each term must contain all the letters and all the subscripts of the determinants (18), all the terms may be written in the natural order of the letters or in the natural order of the subscripts. The former is the more common method, and should be employed by the beginner before determining the sign of the terms (21, 22). When elements

have subscripts $(a_2\ b_3\ c_1\ d_4)$, and the number of inversions of both subscripts and letters are counted, the interchange of ~~an~~ two elements will alter the number of inversions by an even number, since the sum of two odd numbers must be even (6, page 22 ~~18~~). These considerations lead to the important principle: the sign of a term is not affected by commuting its elements (21, 22).

This principle could have been inferred from the fact that determinants result from algebraic elimination; hence the commutative law must hold.

VII.

24. *Number of Terms.* — A determinant of the n^{th} order consists of $n!$ terms.

Demonstration. — All the terms may be formed by keeping the letters in their natural order (23). The first place may be filled in n ways, since there are n different a's; the second in $n-1$ ways. Any one of the n different b's may be chosen except the one whose subscript is the same as the subscript of the a which has been chosen

to fill the first place (18). The third place may be filled in $n-2$ ways, for the same reason, etc.

$$n\,(n-1)\,(n-2)\ \ldots\ 2 \times 1 = n!$$

VIII.

25. *Development by Permuting the Subscripts.* — All the terms of the determinant may be obtained from a given term by keeping the letters in a given order, and permuting the subscripts (24, 18).

IX.

26. *Other Methods of Notation.* — Many different methods of writing a determinant are employed. Some will be explained later. The following abbreviated methods will be employed in this work.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & \ldots & l_1 \\ a_2 & b_2 & \ldots & l_2 \\ \ldots & \ldots & \ldots & \ldots \\ a_n & b_n & \ldots & l_n \end{vmatrix}$$

will be written respectively

$$(a_1\, b_2),\ (a_1\, b_2\, c_3),\ (a_1\, b_2\ \ldots\ l_n), \text{ or}$$

$$\Sigma \pm (a_1\, b_2),\ \Sigma \pm (a_1\, b_2\, c_3),\ \Sigma \pm (a_1\, b_2\ \ldots\ l_n);$$

i.e., only the Principal Diagonal will be given. All the other terms can easily be obtained from it (25). We shall also frequently employ the Greek letter Δ to denote a determinant.

The learner should make himself quite familiar with the preceding definitions and deductions, since they involve almost the entire theory of elementary determinants. We shall now proceed to the study of separate determinants, beginning with the simplest form.

27. *Determinants of the Second Order.*—

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1\, b_2 - a_2\, b_1.$$

QUERIES.

1. How are the terms identical with $(a_1\, b_2)$ obtained? (18, 25, 24.)

2. What determines the signs? (21, 22.)

3. Do all determinants of the second order represent two terms? (18, 25.)

4. If one of the elements is 0, what is the value of one of the terms ?

5. Explain when two elements become 0's. (Two cases.)

6. When three elements become 0's.

7. Does the order of the factors of a term affect the value of the term ? (23.)

8. When is the value of a determinant of the second order negative ?

9. When will the two terms have the same sign ?

10. If the two rows or the two columns of $(a_1\ b_2)$ are identical, what is its value ?

11. When the rows in order are made the columns in order, is the value of the determinant affected ?

12. Is the value of the determinant altered by interchanging the two rows ? the two columns ?

13. What effect does the multiplication of a line of $(a_1\ b_2)$ have on the value of the determinant ?

28. *Direction.*—Find the value of the following ~~numerical~~ determinants.

1–3. $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$, $\begin{vmatrix} 6 & 8 \\ 2 & 9 \end{vmatrix}$, $\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix}$.

Ans. 2, 38, 14.

4–5. $\begin{vmatrix} -4 & 1 \\ -2 & -3 \end{vmatrix}$, $\begin{vmatrix} -4 & -7 \\ -3 & -5 \end{vmatrix}$.

Ans. 14, -1.

6–8. $\begin{vmatrix} a^2 & b^2 \\ c & d \end{vmatrix}$, $\begin{vmatrix} abc & -d \\ d & b \end{vmatrix}$, $\begin{vmatrix} 0 & -3 \\ 0 & 10 \end{vmatrix}$.

Ans. $a^2d - b^2c$, $ab^2c + d^2$, 0.

9–10. $\begin{vmatrix} 4 & 0 \\ 0 & 8 \end{vmatrix}$, $\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$.

Ans. 32, 4.

11–12. $\begin{vmatrix} \frac{1}{5} & 4 \\ 3 & 1 \end{vmatrix}$, $\begin{vmatrix} -\frac{1}{2} & 0 \\ .2 & 6 \end{vmatrix}$,

13–14. $\begin{vmatrix} .7 & 2 \\ .3 & -.5 \end{vmatrix}$, $\begin{vmatrix} 4 & a \\ 2 & -3 \end{vmatrix}$.

15–16. $\begin{vmatrix} 1\frac{1}{2} & 3 \\ 4 & 0 \end{vmatrix}$, $\begin{vmatrix} y & a \\ z & b \end{vmatrix}$.

17–18. $\begin{vmatrix} \frac{1}{4} & \frac{1}{x} \\ -3 & x \end{vmatrix}$, $\begin{vmatrix} a-b & d \\ c^2 & y \end{vmatrix}$.

29. *Determinants of the Third Order.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\equiv \begin{array}{l} \quad a_1\, b_2\, c_3 - a_1\, b_3\, c_2 + a_2\, b_3\, c_1 \\ - a_2\, b_1\, c_3 + a_3\, b_1\, c_2 - a_3\, b_2\, c_1. \end{array}$$

QUERIES.

1. How may the terms identical with $(a_1\, b_2\, c_3)$ be obtained? (18, 25, 24.)

2. Give the reasons for the signs of the terms (21, 22).

3. Determine the signs of the following terms in three ways (21, 22, 23), and compare the results.

$c_1b_3a_2,\quad a_2b_1c_3,\quad a_3c_2b_1,\quad b_2c_3a_1,\quad b_3a_1c_2.$

4. If h and i represent the number of inversions of letters and subscripts respectively, $(-1)^{i+h}$ will determine the sign of a term, and is generally called the sign factor of the term. Explain.

30. An easy method for developing a determinant of the third order is explained on page 18. The learner should become very familiar with this method, since it is the one most commonly employed.

31. Instead of connecting the terms as indicated on page 18, two of the columns may be repeated thus,

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

The products of elements in the Principal Diagonal and in the two lines parallel to it which contain three elements constitute the positive terms. The negative terms are found similarly with respect to the Secondary Diagonal.

32. The development may be effected directly from definitions (18, 22) as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Explanation. — Since no two elements of a term belong to the same row or column (18), the only elements that may be combined with a_2 are b_1, c_1, b_3, c_3. Hence the only two terms containing a_2 are $a_2\, b_1\, c_3$ and $a_2\, b_3\, c_1$. We find the terms which contain a_1 and those which contain a_3 in a similar manner. The sum of these terms with the proper signs constitutes the development of Δ.

Queries.

1. What are the terms which contain a_1?
2. What terms contain a_3?
3. Why is the sum of the terms containing a_1, a_2, a_3 the development of Δ?

Examples.

33. *Direction.* — Find the values of the following determinants of the third order. Use the method explained on page 18.

1. $$\begin{vmatrix} 2 & -4 & -3 \\ 5 & 1 & 2 \\ 3 & 0 & 5 \end{vmatrix}.$$

Solution. —

$$\begin{aligned} 2 \times 1 \times 5 &= 10, \\ 5 \times 0 \times -3 &= 0, \\ 3 \times -4 \times 2 &= -24, \\ &\overline{-14} \end{aligned}$$

$$\begin{aligned} 3 \times 1 \times -3 &= -9, \\ 5 \times -4 \times 5 &= -100, \\ 2 \times 0 \times 2 &= 0, \\ &\overline{-109.} \end{aligned}$$

$-14 - (-109) = 95$, the value of Δ.

2–4. $$\begin{vmatrix} 4 & 2 & 0 \\ 6 & 3 & 2 \\ 1 & 2 & 0 \end{vmatrix}, \quad \begin{vmatrix} 3 & 7 & 1 \\ 4 & 3 & 2 \\ 1 & 5 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 7 & 1 \\ 0 & 2 & 4 \\ 0 & 3 & 5 \end{vmatrix}.$$

Ans. $-12, 1, 0.$

5–6. $\begin{vmatrix} 4 & 5 & 7 \\ 3 & 2 & 6 \\ 8 & 10 & 14 \end{vmatrix}$, $\begin{vmatrix} x & -y & 3 \\ 4 & 3 & -1 \\ -x & 2 & 0 \end{vmatrix}$.

Ans. $0,\ 11x - xy + 24$.

7. $\begin{vmatrix} 4 & -\frac{1}{5} & 3 \\ -\frac{2}{3} & 6 & 2 \\ 5 & -3 & 15 \end{vmatrix}$.

Ans. 296.

What general principle may be deduced from **4**?

Prove by expanding

8. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

9. $\begin{vmatrix} 2 & -7 & 4 \\ 3 & 6 & 1 \\ 0 & 2 & -1 \end{vmatrix} \equiv \begin{vmatrix} 2 & 3 & 0 \\ -7 & 6 & 2 \\ 4 & 1 & -1 \end{vmatrix}$.

What principle may be inferred from **8** and **9**? (Compare with **11**, page 39).

Prove by expanding

10. $\begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1x & b_1x & c_1x \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2x & b_2x & c_2x \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

11. $$x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1x \\ a_2 & b_2 & c_2x \\ a_3 & b_3 & c_3x \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}.$$

12. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}.$$

13. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + 2\,b_1 & b_1 & c_1 \\ a_2 + 2\,b_2 & b_2 & c_2 \\ a_3 + 2\,b_3 & b_3 & c_3 \end{vmatrix},$$

14. $$\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} = 0.$$

NOTE. — In what follows Δ will be used to represent the general determinant before and Δ' to represent it after transformation.

34. *Rows and Columns Interchanged.* — If the rows in order are made the columns in order, or *vice versa*, the value of the determinant is not altered; i.e., $\Delta = \Delta'$.

Demonstration. — Denoting by Δ and Δ' respectively the original determinant, and the determinant after the rows in order have been made the columns in order, or the columns in order have been made the rows in order, then will $\Delta \equiv \Delta'$.

Terms of Δ formed with respect to the columns are identical with terms similarly formed from Δ' with respect to rows. The signs of these terms are the same; for if $a, b, c \ldots$ represent the numbers of the columns, and $l, m, n \ldots$ represent the numbers of the rows of Δ from which the elements of any term have been chosen, then will $l, m, n \ldots$ represent the numbers of the columns, and $a, b, c \ldots$ represent the numbers of the rows of Δ' from which the elements of the identical term have been chosen. Therefore all the identical terms have the same sign (21), and $\Delta = \Delta'$. (See 33, **8** and **9**.)

35. *Rows or Columns Mutually Interchanged.* — The interchange of any two rows or of any two columns changes the sign, but not the absolute value of the determinant.

Demonstration. — The terms chosen similarly from Δ and Δ' differ only with respect to two subscripts or two letters, which must be interchanged in all of the terms to make them identical. Therefore all of the terms similarly chosen have different signs, and $\Delta = -\Delta'$. (6, 21, 22), (33, **12**).

36. *Identical Lines.* — When two rows or two columns are identical, the determinant equals 0.

Demonstration. — Interchange the identical lines. This will not alter the determinant. $\therefore\ \Delta = \Delta'$. By (35), $\Delta = -\Delta'$. Hence $\Delta' = -\Delta'$, ~~and $\Delta = -\Delta'$~~. When a change of sign does not affect the value of a quantity, it must be 0. (33, **14**.)

37. *Multiplication of Lines.* — Multiplying or dividing a row or a column of a determinant multiplies or divides the determinant by the factor.

Demonstration. — Each term of Δ' contains the given factor, or is divided by it (18). Terms similarly formed from Δ and Δ' are made identical by the removal of

this factor. $\therefore$ $\Delta' = \Delta \times$ or $\div$ by the factor. (33, **10**, **11**.)

Corollary.—If the elements of a line are equal multiples of the elements of a parallel line, the value of the determinant is 0.

Suggestion.—Divide by a factor which will make the elements identical. Then consult 36.

38. *Transposition of an Element.*—Any element of a determinant may be transposed to any desired place.

Demonstration.—The element may be brought to the required row by interchanging two rows, then to the required column by interchanging two columns (35). When these two interchangings are required, the sign of the determinant will not be affected, since the sign has been changed twice.

It is more common to effect this transformation by transposing the lines to the required places without affecting the relative positions of the other lines. This can

obviously be effected by successive interchanges of adjacent parallel lines. The sign of the resulting determinant is determined by $(-1)^{r+c}$, where $\begin{cases} r \\ c \end{cases}$ represents the number of $\begin{cases} \text{rows} \\ \text{columns} \end{cases}$ over which the line containing the given element has been transposed (8).

Exercise.

39. *Direction.* — Consult first the articles to which reference is made. Then prove the equality by solving the numerical determinants.

34. $$\begin{vmatrix} 2 & 3 & 7 \\ 4 & 0 & 0 \\ 1 & -5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 1 \\ 3 & 0 & -5 \\ 7 & 0 & 6 \end{vmatrix}.$$

35. $$\begin{vmatrix} 1 & 3 & 1 \\ 0 & 5 & -3 \\ 2 & -2 & 3 \end{vmatrix}, = -\begin{vmatrix} 1 & 1 & 3 \\ 0 & -3 & 5 \\ 2 & 3 & -2 \end{vmatrix},$$

$$= -\begin{vmatrix} 0 & 5 & -3 \\ 1 & 3 & 1 \\ 2 & -2 & 3 \end{vmatrix}.$$

$$36. \quad \begin{vmatrix} 2 & 5 & 2 \\ 5 & 0 & 5 \\ -3 & 3 & -3 \end{vmatrix} = 0.$$

$$37. \cdot \textit{Cor.} \quad \begin{vmatrix} 5 & 3 & 6 \\ 2 & 4 & 8 \\ 5 & 10 & 20 \end{vmatrix} = 0.$$

$$37. \quad \begin{vmatrix} 12 & 4 & 1 \\ 6 & 1 & 2 \\ 18 & 0 & 1 \end{vmatrix}, = 6 \begin{vmatrix} 2 & 4 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix}, = \begin{vmatrix} 2 & 4 & 1 \\ 6 & 6 & 12 \\ 3 & 0 & 1 \end{vmatrix}.$$

the following

The element 7 in Δ is transposed to the upper left-hand corner in Δ', and to the upper right-hand corner in Δ''. Explain the signs and prove the equality.

$$38. \quad \begin{vmatrix} 5 & 1 & 0 \\ 3 & 7 & 2 \\ 1 & -4 & -3 \end{vmatrix} = \begin{vmatrix} 7 & 3 & 2 \\ 1 & 5 & 0 \\ -4 & 1 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & 7 \\ 5 & 0 & 1 \\ 1 & -3 & -4 \end{vmatrix}.$$

1. Explain the difference by illustrations of interchanging elements or lines,

and transposing (transferring, moving) an element or a line over a given number of elements or lines. (6, 8, 34, 38).

2. Find the values of the following by multiplying rows or columns to reduce to a simpler form, before evaluating the determinants.

$$(a).\begin{vmatrix} 6 & 3 & 2 \\ 4 & -6 & 8 \\ 2 & 3 & 4 \end{vmatrix}, \qquad (b).\begin{vmatrix} \frac{1}{2} & -\frac{3}{4} & 1 \\ 2 & \frac{1}{2} & 3 \\ \frac{1}{4} & 5 & 0 \end{vmatrix},$$

$$(c).\begin{vmatrix} a^2 & b^2 & 0 \\ ac & bd & c \\ a^2 & ab & 0 \end{vmatrix}.$$

Solution. —

$$\begin{vmatrix} 6 & 3 & 2 \\ 4 & -6 & 8 \\ 2 & 3 & 4 \end{vmatrix} = 2.\,3.\,2\begin{vmatrix} 3 & 1 & 1 \\ 2 & -2 & 4 \\ 1 & 1 & 2 \end{vmatrix} =$$

$$12.\,2\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 24\,[-6+1+2+1-2-6] = -240.$$

3. Do all the principles which apply to rows apply to columns? (34.)

4. Write the terms of $(a_1\ b_2\ c_3\ d_4)$ which contain both a_3 and c_2. (26, 25, 18, 32.)

5. Write the terms of $(a_1\ b_2\ c_3\ d_4\ e_5)$ which contain $a_2\ b_3\ c_1$. Also those which contain $c_2\ b_5$.

6. Of how many terms are the two preceeding determinants composed? (24.)

7. Show that in a determinant of the n^{th} order only two terms have $n-2$ elements in common and that these have opposite signs. (18, 22.)

8. In the determinant $(a_1\ b_2\ c_3\ .\ .\ .\ l_n)$ (page 37) $a\,!$ terms have $n-a$ elements in common, and $(n-a)\,!$ have a elements in common. Explain.

9. How is the value of a determinant affected by changing the signs of all the elements in n rows? (37.)

Prove that

10. $$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} bcd & 1 & a & a^2 \\ acd & 1 & b & b^2 \\ abd & 1 & c & c^2 \\ abc & 1 & d & d^2 \end{vmatrix}.$$

Suggestion. — Multiply the first column by $abcd$, then divide the rows by different factors.

$$\textbf{11.}\ \begin{vmatrix} a_1 + b_1 & c_1 & d_1 \\ a_2 + b_2 & c_2 & d_2 \\ a_3 + b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

40. If all of the elements of a line of a determinant are binomials, $\Delta = \Delta' + \Delta''$, where $\Delta' = \Delta$ with the first terms of the binomial as elements in place of the binomial elements, $\Delta'' = \Delta$ with the second terms of the binomials as elements in place of the binomial elements.

Demonstration. — Each term of Δ has a binomial factor (18). Decompose each term into two terms with monomial factors; e.g., example 11 is equal to $(a_1 + b_1)\, c_2\, d_3 - (a_3 + b_3)\, c_2\, d_1 + (a_2 + b_2)\, c_3\, d_1 -$ etc. $=$ $a_1\, c_2\, d_3 - a_3\, c_2\, d_1 + a_2\, c_3\, d_1 -$ etc., which are terms of Δ' plus $b_1\, c_2\, d_3 - b_3\, c_2\, d_1 + b_2\, c_3\, d_1 -$ etc., which are terms of Δ''. Separate all the resulting terms into two groups — the first containing the first terms

of the binomials, and the second, the second terms of the binomials. Then will the first group equal Δ' and the second Δ''. The reasoning will become clear by completing the solution of this example.

Scholium. — The same reasoning may be employed when a line contains polynomials of any given form or of different forms. Also when the elements in several or all of the parallel lines are polynomials.

1. Prove that

$$\begin{vmatrix} 3+2 & 2+3+5 & 1 \\ 4-1 & 2-6 & 2 \\ 5-4 & 4+2-3 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 2+3+5 & 1 \\ 4 & 2-6 & 2 \\ 5 & 4+2-3 & 3 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & 2+3+5 & 1 \\ -1 & 2-6 & 2 \\ -4 & 4+2-3 & 3 \end{vmatrix}.$$

2. Resolve Δ' and Δ'' each into two determinants, and prove the sum of the four $= \Delta$.

41. A determinant is unaltered by adding to the elements of a line equal multiples of the corresponding elements of a parallel line.

Demonstration. — $\Delta' = \Delta + \Delta''$ (40). $\Delta'' = 0$ (37 *Cor.*). This principle is very important. We shall therefore apply it to a general determinant, to enable the learner to understand the demonstration better.

To prove

$$\begin{vmatrix} a_1 & b_1 & \dots & k_1 & \dots & l_1 & \dots \\ a_2 & b_2 & \dots & k_2 & \dots & l_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_r & b_r & \dots & k_r & \dots & l_r & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_s & b_s & \dots & k_s & \dots & l_s & \dots \end{vmatrix} =$$

$$\begin{vmatrix} a_1 & b_1 & \dots & k_1 + ml_1 & \dots & l_1 & \dots \\ a_2 & b_2 & \dots & k_2 + ml_2 & \dots & l_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_r & b_r & \dots & k_r + ml_r & \dots & l_r & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_s & b_s & \dots & k_s + ml_s & \dots & l_s & \dots \end{vmatrix}.$$

m represents any number, and $k_1, k_2, \dots, k_r, \dots, k_s, \dots$, and $l_1, l_2, \dots, l_r, \dots$, and l_s are the elements of any two columns. Any number of columns may precede, be intermediate, or follow these two columns. By 40 ~~Δ' is equal to~~ $\Delta' + \Delta''$ are equal to

$$
\begin{vmatrix}
a_1 & b_1 & \dots & k_1 & \dots & l_1 & \dots \\
a_2 & b_2 & \dots & k_2 & \dots & l_2 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_r & b_r & \dots & k_r & \dots & l_r & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_s & b_s & \dots & k_s & \dots & l_s & \dots
\end{vmatrix}
+
$$

$$
\begin{vmatrix}
a_1 & b_1 & \dots & ml_1 & \dots & l_1 & \dots \\
a_2 & b_2 & \dots & ml_2 & \dots & l_2 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_r & b_r & \dots & ml_r & \dots & l_r & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_s & b_s & \dots & ml_s & \dots & l_s & \dots
\end{vmatrix}.
$$

(Δ') is equal to Δ;

The first of these determinants is Δ, and the second (Δ'') is equal to 0. $\therefore\ \Delta' + \Delta'' = \Delta$.

Q. E. D.

Cor.—A determinant is unaltered by adding to the elements of one line equal multiples of the corresponding elements of two or more parallel lines.

Scholium.—Denoting by $l_1, l_2, l_3, l_4, \dots$ the parallel lines of a determinant, the determinant is not altered by writing instead, $l_1 + ml_3$, $l_2 + nl_3 + l_4$, $l_3 - l_4$, $l_4, \dots$: but we cannot substitute $l_1 + l_2$, $l_2 + l_1$, l_3, $l_4, \dots$ because after $l_1 + l_2$ was substituted for l_1, the lines were $l_1 + l_2$, l_2, l_3, $l_4 \dots$

We have therefore no reason to suppose that the addition of l_1 will not alter the determinant. In the case supposed $\Delta' = 0$ (36). It is evident that we could have added the transformed line $l_1 + l_2$, giving $l_1 + l_2,\ 2\,l_2 + l_1,\ l_3,\ l_4$.

This principle is very useful in the reduction of numerical determinants. The following examples will show how the principle is applied.

To find the value of the following determinant we divide the first

$$\begin{vmatrix} 4 & 6 & 8 \\ 7 & 3 & 5 \\ 2 & -3 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 & 4 \\ 5 & 0 & 1 \\ 0 & -6 & 0 \end{vmatrix} = -216.$$

row by 2, and subtract the quotient from the second and third rows.

Direction. — Find the values of the following by reducing according to 41.

$$\begin{vmatrix} 3 & 4 & 6 \\ 1 & -6 & 2 \\ 5 & 7 & 11 \end{vmatrix}, \quad \begin{vmatrix} 8 & 7 & 3 \\ 4 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix}, \quad \begin{vmatrix} -8 & 5 & -1 \\ 1 & 2 & 3 \\ -4 & 2\frac{1}{2} & -\frac{1}{2} \end{vmatrix}.$$

42. If all the elements except one of a line are zeros, the determinant is equal to

the determinant formed by striking out the lines which contain the significant element, multiplied by $(-1)^{r+c}$ times the significant element; e.g.,

$$\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}.$$

Demonstration. — Since all the terms of Δ must contain one element from the line in which there is only one significant element (18), all the terms which do not reduce to zeros must contain this element. The elements which unite with any given element in a determinant are found by permuting the other subscripts with the other letters; for all the permutations of 1, 2, 3, 4, . . . which may be formed by keeping 1 in its place are found by prefixing 1 to all the permutations of 2, 3, 4, . . . (See 18, 23, 25); but permuting the other subscripts with the other letters will evidently lead to the determinant formed by striking out the row and column containing the given element.

If the significant element is in the

upper left-hand corner the sign is positive. It may always be transposed to that place without altering the value of the determinant by prefixing the sign factor $(-1)^{r+c}$ (38); i.e., $\Delta = (-1)^{r+c}\,\Delta'$.

Again $(-1)^{r+c}$ determines whether an element causes the sign of a term from which it is removed, to change. Therefore, when the significant element is removed from all the terms which do not reduce to zeros and placed before a parenthesis enclosing them, it must be preceded by the sign factor $(-1)^{r+c}$. When the parenthesis is written as a determinant this sign factor remains.

Prove that

1. $$\begin{vmatrix} 2 & -3 & 0 \\ 1 & -4 & -2 \\ 2 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 & 0 \\ 5 & a & -2 \\ 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 4 \\ 0 & 0 & -2 \\ 2 & 1 & 10 \end{vmatrix}$$

Δ' may easily be found by cancelling the row and the column containing the significant element. The given determinant

and those on page 58 should, for this purpose, be written

$$\begin{vmatrix} 2 & -3 & 0 \\ 1 & -4 & -2 \\ 2 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix}.$$

2. When a line contains only one significant element, is the determinant altered by changing the other elements in the line which is perpendicular line to this and containi~~ng~~ the given element? (1.)

43. The order of a determinant may be increased indefinitely, for the borders

$$\begin{array}{llll} 1 & 0 & 0 & \ldots \\ x & & & \\ y & & & \\ \vdots & & & \end{array} \qquad \begin{array}{llll} 1 & x & y & \ldots \\ 0 & & & \\ 0 & & & \\ \vdots & & & \end{array} \qquad \begin{array}{llll} 1 & 0 & 0 & \ldots \\ 0 & & & \\ 0 & & & \\ \vdots & & & \end{array}$$

may be placed on any determinant without altering the value.

Prove that

1. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ c_2 & a_1 & b_1 \\ c_3 & a_2 & b_2 \end{vmatrix}.$

2. $$\begin{vmatrix} 2 & -3 \\ 1 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & -3 \\ 5 & 1 & 11 \end{vmatrix}.$$

3. $$\begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 2 & -1 \end{vmatrix}.$$

4. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + 0 + 0 & b_1 & c_1 \\ 0 + a_2 + 0 & b_2 & c_2 \\ 0 + 0 + a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

(40, *Scholium*; 42.)

5. $$\begin{vmatrix} a - x & a_1 - x \\ b - x & b_1 - x \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & a - x & a_1 - x \\ 0 & b - x & b_1 - x \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & 1 \\ x & a & a_1 \\ x & b & b_1 \end{vmatrix} = x \begin{vmatrix} 0 & 1 & 1 \\ 1 & a & a_1 \\ 1 & b & b_1 \end{vmatrix} + \begin{vmatrix} a & a_1 \\ b & b_1 \end{vmatrix}.$$

44. Any determinant may be so reduced that all the elements except one of a given row are zeros.

Demonstration.— If none of the elements of the given row are zeros, reduce them to their L. C. M. by multiplying the columns; then subtract one of the columns from all the others. The result will be the required determinant. If some of the elements in the given row are already zeros, treat only the significant elements in this way.

Illustration. —

$$\begin{vmatrix} 3 & 4 & a & l \\ b & c & 2 & 1 \\ a & b & 4 & l \\ 1 & 0 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 12al & 12al & 12al & 12al \\ 4abl & 3acl & 24l & 12a \\ 4a^2l & 3abl & 48l & 12al \\ 4al & 0 & 36l & 24a \end{vmatrix} =$$

$$\begin{vmatrix} 12al & 0 & 0 & 0 \\ 4abl & 3acl - 4abl & 24l - 4abl & 12a - 4abl \\ 4a^2l & 3abl - 4a^2l & 48l - 4a^2l & 12al - 4a^2l \\ 4al & -4al & 36l - 4al & 24a - 4al \end{vmatrix}$$

$$= 12al \begin{vmatrix} 3acl - 4abl & 24l - 4abl & 12a - 4abl \\ 3abl - 4a^2l & 48l - 4a^2l & 12al - 4a^2l \\ -4al & 36l - 4al & 24a - 4al \end{vmatrix}.$$

The reduction would have been much simpler if the last row had been chosen for the given row.

45. The order of a determinant may be decreased by the method employed on page 61, problem 4, or according to 44. The judicious use of the principle explained in 41 simplifies these transformations greatly.

EXAMPLES.

1. $$\begin{vmatrix} 9 & 13 & 17 & 4 \\ 18 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 4 & 1 & 8 \\ 4 & 1 & 2 & 13 \\ 2 & 4 & 2 & 11 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 4 & 1 & 2 & 6 \\ 2 & 4 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & -1 & -1 \\ 4 & -3 & -2 & 2 \\ 2 & 2 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 & -1 \\ -3 & -2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & -1 & -1 \\ -7 & -2 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & -1 \\ 1 & -4 \end{vmatrix} = 15.$$

Explanation. — Δ' is obtained by subtracting multiples of the last column from

the others. Δ'' is obtained from Δ' by subtracting the sum of the first three columns from the last, etc.

Direction. — Find the values of

2, 3. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix}$, $\begin{vmatrix} 2 & 0 & 0 & -1 \\ 4 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & b \end{vmatrix}$,

4. $\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix}$.

Ans. 0. $2(5b+2), -972$.

46. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & \dots & k_1 & l_1 \\ a_2 & b_2 & c_2 & \dots & k_2 & l_2 \\ a_3 & b_3 & c_3 & \dots & k_3 & l_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & k_n & l_n \end{vmatrix} =$$

$$\frac{1}{b_1\, c_1 \dots k_1} \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & \dots & \begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \dots & \begin{vmatrix} k_1 & l_1 \\ k_3 & l_3 \end{vmatrix} \\ \dots & \dots & \dots & \dots \\ \begin{vmatrix} a_1 & b_1 \\ a_n & b_n \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_n & c_n \end{vmatrix} & \dots & \begin{vmatrix} k_1 & l_1 \\ k_n & l_n \end{vmatrix} \end{vmatrix}.$$

Hints. — Multiplying the first column by $-b_1$ and the second by a_1 we add the results; then multiply the second by $-c$ and the third by b_1, etc. Operating on all of the columns ($n-1$ times) in this way we obtain.

$$(-1)^{n-1}\, b_1\, c_1 \ldots k_1\, l_1\, \Delta =$$

$$\begin{vmatrix} 0 & 0 & \ldots & 0 & l_1 \\ -a_2 b_1 + a_1 b_2 & -b_2 c_1 + b_1 c_2 & \ldots & -k_2 l_1 + k_1 l_2 & l_2 \\ -a_3 b_1 + a_1 b_3 & -b_3 c_1 + b_1 c_3 & \ldots & -k_3 l_1 + k_1 l_3 & l_3 \\ \ldots\quad\ldots & \ldots & \ldots & \ldots & \ldots \\ -a_n b_1 + a_1 b_n & -b_n c_1 + b_1 c_n & \ldots & -k_n l_1 + k_1 l_n & l_n \end{vmatrix}.$$

By making the last column the first the sign factor $(-1)^{n-1}$ will disappear (35), since it may be accomplished by $^{n-1}$ successive interchanges of adjacent columns. We now reduce the determinant to one whose order is $n-1$ (42). Then $b_1\, c_1 \ldots k_1\, l_1\, \Delta = l_1\, \Delta'$ or $\Delta = \dfrac{\Delta'}{b_1\, c_1 \ldots k_1} = \dfrac{1}{b_1\, c_1 \ldots k_1}\, \Delta'$.

Q. E. D.

Scholium. — The simplest line should be made the first row before this method is employed.

Illustrative Solution. —

$$\begin{vmatrix} 0 & -2 & 5 & 3 \\ 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 5 & 2 & 3 & 4 \end{vmatrix} =$$

$$\frac{1}{-2.5}\begin{vmatrix} \begin{vmatrix} 0 & -2 \\ 7 & -2 \end{vmatrix} & \begin{vmatrix} -2 & 5 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 5 & 3 \\ 0 & 5 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ -2 & 6 \end{vmatrix} & \begin{vmatrix} -2 & 5 \\ 6 & -2 \end{vmatrix} & \begin{vmatrix} 5 & 3 \\ -2 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ 5 & 2 \end{vmatrix} & \begin{vmatrix} -2 & 5 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 3 \\ 3 & 4 \end{vmatrix} \end{vmatrix} =$$

$$\frac{1}{-2.5}\begin{vmatrix} 14 & 10 & 25 \\ -4 & -26 & 16 \\ 10 & -16 & 11 \end{vmatrix} =$$

$$\frac{1}{-2.5.10}\begin{vmatrix} \begin{vmatrix} 14 & 10 \\ -4 & -26 \end{vmatrix} & \begin{vmatrix} 10 & 25 \\ -26 & 16 \end{vmatrix} \\ \begin{vmatrix} 14 & 10 \\ 10 & -16 \end{vmatrix} & \begin{vmatrix} 10 & 25 \\ -16 & 11 \end{vmatrix} \end{vmatrix} =$$

$$\frac{-1}{100}\begin{vmatrix} -324 & 810 \\ -324 & 510 \end{vmatrix} =$$

$$\frac{-1}{100}\begin{vmatrix} 0 & 300 \\ -324 & 510 \end{vmatrix} = -972.$$

These reductions could have been simplified by (37). We have not employed many other principles for the purpose of impressing the one under consideration. It should be observed that only the extreme elements in the first row may be zeros. For the reduction of numerical determinants containing few or no zeros we consider this method superior. Introduce a few examples ill. last article see v I.

We have now indicated three general methods by which all determinants may be reduced (43, **4**, 44 and 42, 46) as far as may be desired. Which of these methods should be employed is determined by the nature of the problem. Frequently several methods are used in the solution of one problem. Determinants of higher orders should generally be reduced to the second order before evaluating.

47.

(a). $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$, $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$; $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

(b). $\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$; (c). $\begin{vmatrix} c_1 & d_1 \\ c_4 & d_4 \end{vmatrix}$; (d). d_2

(*b*), (*c*), (*d*) are respectively first, second and third minors of $(a_1\, b_2\, c_3\, d_4)$.

$$(e)\ a_1 \text{ and } \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} : (f) \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\text{and } \begin{vmatrix} c_1 & d_1 \\ & \\ c_4 & d_4 \end{vmatrix} : (g)\ (a_1\ b_3\ c_4) \text{ and } d_2$$

(*e*), (*f*), (*g*) are respectively complementary minors.

48. If we delete one line and one column of a determinant, the remaining elements in their relative positions may be regarded as a determinant. This determinant is called the first or the principal minor. There are n^2 principal minors in a determinant of the n^{th} order.

49. If we delete the same number of rows and columns, the remaining elements in their relative positions constitute a minor whose order is $n - a$, n being the order of the determinant and a the number of the rows deleted. [(*c*), (*d*),].

50. *Definition.* — If we cancel the same number of rows and columns, the twice cancelled and uncancelled elements respec-

tively constitute *complementary minors* of the determinant. When their signs are prefixed they are called *co-factors.* [(*e*), (*f*), (*g*)].

51. The *principal minors*, which are the complements of $a_1, a_2, a_3, \ldots b_1, b_2, b_3, \ldots$ are denoted by $A_1, A_2, A_3, \ldots B_1, B_2, B_3, \ldots$ In (*a*) (47)

$$A_1 = \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}, \qquad A_2 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix},$$

$$A_3 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

$$B_1 = \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ b & c_4 & d_4 \end{vmatrix}, \qquad B_2 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix},$$

$$B_4 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}.$$

It is evident that there are as many principal minors as there are elements, and that any principal minor may be obtained by deleting the lines containing the corresponding element.

52. From 42 and 43, **4**, it may be seen that

$(a_1\, b_2\, c_3\, d_4)$

$$
\begin{aligned}
&= a_1\, A_1 - a_2\, A_2 + a_3\, A_3 - a_4\, A_4. \\
&= - b_1\, B_1 + b_2\, B_2 - b_3\, B_3 + b_4\, B_4. \\
&= c_1\, C_1 - c_2\, C_2 + c_3\, C_3 - c_4\, C_4. \\
&= - d_1\, D_1 + d_2\, D_2 - d_3\, D_3 + d_4\, D_4.
\end{aligned}
$$

Addenda see front I.

Multiplication by Co-factors.

53. If the elements of a line are respectively multiplied by the co-factors of the corresponding elements of a parallel line the sum of the products is 0.

Demonstration. — In the determinant $\Sigma \pm (a_1\, b_2 \ldots l_n)$, let us multiply the column containing the a's and the column containing the b's by the co-factors of the first column, the results are

$$
\begin{vmatrix}
a_1\, A_1 & b_1 & \ldots & l_1 \\
- a_2\, A_2 & b_2 & \ldots & l_2 \\
\ldots & \ldots & \ldots & \ldots \\
\pm\, a_n\, A_n & b_n & \ldots & l_n
\end{vmatrix}
$$

$$
\begin{vmatrix}
a_1 & b_1\, A_1 & \ldots & l_1 \\
a_2 & - b_2\, A_2 & \ldots & l_2 \\
\ldots & \ldots & \ldots & \ldots \\
a_n & \pm\, b_n\, A_n & \ldots & l_n
\end{vmatrix}.
$$

These multiplied columns equal respectively (52),

$$\begin{vmatrix} a_1 & b_1 & \dots & l_1 \\ a_2 & b_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots \\ a_n & b_n & \dots & l_n \end{vmatrix}, \quad \begin{vmatrix} b_1 & b_1 & \dots & l_1 \\ b_2 & b_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots \\ b_n & b_n & \dots & l_n \end{vmatrix}.$$

The first is the original determinant (52), and the second is 0 (36). It should be observed that the second Δ is obtained from the first by writing b_1, b_2, . . . in place of a_1, a_2, . . .

Illustrative Example. —

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & -7 & 1 \\ 3 & 2 & 0 \end{vmatrix}. \quad C_1 = 23, + C_2 = 5, C_3 = -17.$$

Multiplying the first and second columns by these co-factors, we have

$$2 \times 23 + 1 \times 5 - 3 \times 17 = 0.$$
$$3 \times 23 - 7 \times 5 - 2 \times 17 = 0.$$

Application.

54. Given

$$\left.\begin{array}{l} a_1 x + b_1 y + c_1 z = m_1 \\ a_2 x + b_2 y + c_2 z = m_2 \\ a_3 x + b_3 y + c_3 z = m_3 \end{array}\right\} M.$$

to find the values of x, y, and z.

The determinant of the system is $(a_1 b_2 c_3)$. Multiplying by x, we obtain

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix} =$$

$$\begin{vmatrix} a_1 x + b_1 y + c_1 z & b_1 & c_1 \\ a_2 x + b_2 y + c_2 z & b_2 & c_2 \\ a_3 x + b_3 y + c_3 z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}.$$

Hence

$$x = \left| \frac{\begin{matrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{matrix}}{\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}} \right|.$$

We could evidently obtain the values of y and z in a similar way. This method is general. The equations may also be solved in the following way.

We multiply the equations respectively by A_1, $-A_2$, and A_3. The resulting equations are

$$\left.\begin{aligned} a_1 A_1 x + b_1 A_1 y + c_1 A_1 z &= m_1 A_1 \\ -a_2 A_2 x - b_2 A_2 y - c_2 A_2 z &= -m_2 A_2 \\ a_3 A_3 x + b_3 A_3 y + c_3 A_3 z &= m_3 A_3 \end{aligned}\right\} M'.$$

Since any numbers written in the square form may be regarded as a determinant (18), we may consider the co-efficients of the unknowns in M to be a determinant. The notation employed gives it the form of the general determinant of the third order $(a_1\ b_2\ c_3)$.

Assigning the values to A_1, $-A_2$, A_3, which they have in $(a_1\ b_2\ c_3)$, and adding the equations M′ together, we obtain (53)

$$(a_1\,A_1 - a_2\,A_2 + a_3\,A_3)\,x = m_1\,A_1 - m_2\,A_2 + m_3\,A_3.$$

When written in the more common form, this becomes

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}.$$

Hence

$$x = \frac{\begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

See pages 7–18.

55. To find the value of y we multiply equations M by $-B_1$, B_2, and $-B_3$, and add the three equations together. To find the value of z we multiply by C_1, $-C_2$, and C_3, and add the resulting equations. The results will be (53)

$$(-b_1\ B_1 + b_2\ B_2 - b_3\ B_3)\ y = -m_1\ B_1 + m_2 B_2 - m_3 B_3.$$

$$(c_1\ C_1 - c_2\ C_2 + c_3\ C_3)\ z = m_1\ C_1 - m_2\ C_2 + m_3 C_3.$$

Dividing by the co-efficients of these unknowns, and writing the results in the ordinary way, we find the values of the three unknowns to be

$$x = \frac{\begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & m_1 \\ a_2 & b_2 & m_2 \\ a_3 & b_3 & m_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Illustrative Solution. —

Required, the values of x, y, and z in

$$\left.\begin{array}{r} 2x+3y-z=11 \\ x-2y+4z=3 \\ 3y-2z=4 \end{array}\right\}$$

$$x = \frac{\begin{vmatrix} 11 & 3 & -1 \\ 3 & -2 & 4 \\ 4 & 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \\ 0 & 3 & -2 \end{vmatrix}} = \frac{\begin{vmatrix} -31 & 10 \\ 21 & -3 \end{vmatrix}}{\begin{vmatrix} -7 & 10 \\ 6 & -3 \end{vmatrix}} = 3.$$

Explanation. — The reduction has been effected according to 46. Since both numerator and denominator are multiplied by 3, it is omitted.

56. Since the denominator of the fractions representing the values of x, y and z respectively is the same, we need to find its value for but one of the unknowns. We therefore find the values of the other numerators and divide them by the common denominator -13. Thus

$$y = \frac{\begin{vmatrix} 2 & 11 & -1 \\ 1 & 3 & 4 \\ 0 & 4 & -2 \end{vmatrix}}{-13} = \frac{\begin{vmatrix} -5 & 47 \\ 8 & -18 \end{vmatrix}}{-13 \times 11} = 2 : z =$$

$$\frac{\begin{vmatrix} 2 & 3 & 11 \\ 1 & -2 & 3 \\ 0 & 3 & 4 \end{vmatrix}}{-13} = \frac{\begin{vmatrix} -7 & 31 \\ 6 & -21 \end{vmatrix}}{-13 \times 3} = 1.$$

It is not difficult to see that the method pursued in articles 54 and 55 is independent of the number of equations. This gives rise to the following practical rule for the solution of simple simultaneous equations.

The values of the unknowns are represented by fractions whose common denominator is the determinant formed by writing the coefficients of the unknowns in their relative positions. The numerators are obtained from the common denominator by replacing the coefficients of the required unknown by the second members in order.

EXAMPLES.

Direction. — Find the values of the unknowns in the following groups of equations:

$$1. \begin{cases} cx + y = d. \\ 2x - ly = m. \end{cases}$$

Suggestion. — In 1

$$x = \frac{\begin{vmatrix} d & 1 \\ m & -l \end{vmatrix}}{\begin{vmatrix} c & 1 \\ 2 & -l \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} c & d \\ 2 & m \end{vmatrix}}{\begin{vmatrix} c & 1 \\ 2 & -l \end{vmatrix}}.$$

2. $\begin{cases} 3x - 2y = 10. \\ 5x + 4y = 18. \end{cases}$

3. $\begin{cases} 2x + 4y - 3z = 22. \\ 4x - 2y + 5z = 18. \\ 6x + 7y - z = 63. \end{cases}$

4. $\begin{cases} x - 3z + 2y = -26. \\ 2y + x - z = 14. \\ y + 2z - 3x = 22. \end{cases}$

Suggestion. — Always consider the unknowns in the same order in all the equations when writing the determinants. In 4

$$x = \frac{\begin{vmatrix} -26 & 2 & -3 \\ 14 & 2 & -1 \\ 22 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 1 & 2 & -1 \\ -3 & 1 & 2 \end{vmatrix}} \quad y = \frac{\begin{vmatrix} 1 & -26 & -3 \\ 1 & 14 & -1 \\ -3 & 22 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 1 & 2 & -1 \\ -3 & 1 & 2 \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} 1 & 2 & -26 \\ 1 & 2 & 14 \\ -3 & 1 & 22 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 1 & 2 & -1 \\ -3 & 1 & 2 \end{vmatrix}}$$

When one line has a common factor, as -26, 14, 22 in 4, it may be placed before

~~containing this line~~

~~all~~ the determinant~~s~~ and the other factors' placed instead when the determinant~~s~~ ~~are~~ is first written. In this way the operation of reduction may often be greatly abridged.

5. $\begin{cases} 2x + y + 3z = 3. \\ x - 2y = -3\frac{1}{2}. \\ y - 4z = 2. \end{cases}$

6. $\begin{cases} 3x + 9y^{\frac{1}{2}} + 8z = 41. \\ 5x + 4y^{\frac{1}{2}} - 2z = 20. \\ 11x + 7y^{\frac{1}{2}} - 6z = 37. \end{cases}$

7. $\begin{cases} \frac{3}{4}x + \frac{2}{3}y - z = -26. \\ x - \frac{1}{5}y + \frac{3}{5}z = 24. \\ 3x + y + z = 46. \end{cases}$

8. $\begin{cases} x + y + z = a + b + c. \\ cx + ay + bz = a^2 + b^2 + c^2. \\ bx + cy + az = cx + ay + bz. \end{cases}$

Ans. $b + c - a,\ a + c - b,\ a + b - c.$

9. $\begin{cases} 2x = u + y + z. \\ 3y = u + x + z. \\ 4z = u + x + y. \\ u = x - 14. \end{cases}$

10. $\begin{cases} 4x + 4y + z = 38. \\ 3z + 2y - 3t = 4. \\ x - y + 2t = 14. \\ 3z - 2t + u = 9. \\ 2t + y = 21 \end{cases}$

Hints. — Transposing the unknowns in 9 to the first members and arranging in the order x, y, z, u, we find that

$$x = \frac{\begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & -3 & 1 & 1 \\ 0 & 1 & -4 & 1 \\ 14 & 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -4 & 1 \\ 1 & 0 & 0 & -1 \end{vmatrix}} =$$

$$\frac{-14\begin{vmatrix} 0 & 0 & 1 \\ -4 & 0 & 1 \\ 0 & -5 & 1 \end{vmatrix}}{-\begin{vmatrix} -3 & 4 & 0 \\ 0 & -4 & 5 \\ 2 & 1 & -4 \end{vmatrix}} = \frac{-280}{-7} = 40.$$

The reduction has been effected according to 41, 42. When the determinant of the third order contains a number of zeros it is unnecessary to reduce it to a lower order, since it may easily be evaluated. In transforming determinants care should be taken to transform them in such a way as to lead to the smallest possible elements.

$$11. \begin{cases} 2x + y + z = 17. \\ x + y + u = 11. \\ x + 2z + u = 15. \\ y + z + u = 9. \end{cases}$$

$$12. \begin{cases} 2ax + by - cz = 2a^2 + b^2 - c^2. \\ ax - 2by + cz = a^2 - 2b^2 + c^2. \\ ax - by - cz = a^2 - b^2 - c^2. \end{cases}$$

Addenda see II.

Consistence of Equations.

57. *Definition.* — When the number of independent equations is equal to the number of unknowns, the equations are said to be *consistent.*

58. *Definition.* — When there is one more equation than unknowns, the equations are *inconsistent* unless one of the equations is dependent.

59. If a group of n simple equations involving $n - 1$ unknowns is consistent the determinant formed by writing the coefficients of the unknowns and the absolute terms in order equals 0.

Demonstration. — Writing the absolute terms in the first members and denoting

-1 by e, the general group in question becomes

$$\left.\begin{array}{l} a_1\, x + b_1\, y + \ldots + l_1\, z + m_1\, e = 0. \\ a_2\, x + b_2\, y + \ldots + l_2\, z + m_2\, e = 0. \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad = \ldots \\ a_n\, x + b_n\, y + \ldots + l_n\, z + m_n\, e = 0. \end{array}\right\} \text{K.}$$

Since all of these equations are satisfied when $e = -1$ by hypothesis, they must evidently be satisfiable when e is regarded as unknown. Considering e unknown and finding the values of the other unknowns by 56, we find that the determinants of the numerators reduce to 0's, since one of the columns (the absolute terms) consists of zeros: therefore the common denominator must equal zero, otherwise the unknowns could have no value except zero. Q. E. D.

60. If the value of a determinant vanishes, at least one of its rows is dependent.[1]

Demonstration. — To a certain arbitrary multiple of the elements of the second column add arbitrary multiples of the corresponding elements of all the other columns. The factors employed shall all be

[1] See Note A at the end.

finite, greater or less than zero, and be so chosen that none of the sums of the multiples be zero. The determinants may be written as follows: —

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & .. & l_1 \\ a_2 & b_2 & .. & l_2 \\ .. & .. & .. & .. \\ a_n & b_n & .. & l_n \end{vmatrix} = 0$$

$$\Delta' \equiv \begin{vmatrix} a_1 & s_1 & .. & l_1 \\ a_2 & s_2 & .. & l_2 \\ .. & .. & .. & .. \\ a_n & s_n & .. & l_n \end{vmatrix} = 0$$

$s_1\, s_2 \ldots$ represent the sums of the multiples. Hence

$$\Delta' \equiv s_1\, S_1 - s_2\, S_2 + \ldots \pm s_n\, S_n = 0.$$

Therefore s_1 is expressible in terms of s_2, $s_3 \ldots$, which proves the theorem.

The above proof may seem to fail when all the principal minors, S_1, $S_2 \ldots$ are zero. In this case the minors are treated in the same way as the original determinant; and by repeating this process we shall see that either all of the elements in a column are zeros (which would prove

the theorem), or that one line is expressible in terms of the others, which would again be a proof of the theorem.

From the foregoing it follows directly that if the determinant formed by writing the coefficients and absolute terms of n equations containing $n - 1$ unknowns in order vanishes, the equations must be consistent.

Illustrative Solution. —

$$\textbf{1.} \quad \begin{cases} 3x + 4y + 2z = 10. \\ x - 5y + 3z = 12. \\ 4x - y - z = 8. \\ 6x + 8y - 2z = 6. \end{cases}$$

$$\begin{vmatrix} 3 & 4 & 2 & 10 \\ 1 & -5 & 3 & 12 \\ 4 & -1 & -1 & 8 \\ 6 & 8 & -2 & 6 \end{vmatrix} = 0.$$

For if, in the given determinant, the third row be added to the first, and the second to the fourth, the determinant will have two identical lines (41, 36). If we find the values of x, y, and z in any three equations, and substitute in the fourth, the equation will be satisfied (59). E.g.,

$$\begin{vmatrix} 10 & 4 & 2 \\ 8 & -1 & -1 \\ 6 & 8 & -2 \end{vmatrix} - 5 \begin{vmatrix} 3 & 10 & 2 \\ 4 & 8 & -1 \\ 6 & 6 & -2 \end{vmatrix}$$

$$+3 \begin{vmatrix} 3 & 4 & 10 \\ 4 & -1 & 8 \\ 6 & 8 & 6 \end{vmatrix} = 12 \begin{vmatrix} 3 & 4 & 2 \\ 4 & -1 & -1 \\ 6 & 8 & -2 \end{vmatrix}.$$

Combining the first and second, and the third and fourth, we obtain (40)

$$2 \begin{vmatrix} -19 & 5 & 2 \\ -19 & 4 & -1 \\ -38 & 3 & -2 \end{vmatrix} = 6 \begin{vmatrix} 3 & 4 & -1 \\ 4 & -1 & -6 \\ 6 & 8 & -7 \end{vmatrix}.$$

$$-\frac{38}{5} \begin{vmatrix} -1 & -13 \\ -7 & -16 \end{vmatrix} = -\frac{6}{4} \begin{vmatrix} -19 & 25 \\ 0 & 20 \end{vmatrix}.$$

61. *Direction.* — Find which of the following groups of equations are consistent. Verify your results by substituting the values of the unknowns.

2. $\begin{cases} 2x + 3y = 10. \\ 6x - 5y = -12. \\ x + 6y = 15. \end{cases}$

3. $\begin{cases} x - y = 4. \\ 2x + 3y = 9. \\ 5x + 3y = 30. \end{cases}$

4. $\begin{cases} 3x - 2y + 5z = 4. \\ -x + 4y + 3z = 8. \\ 2x - y + 2z = 6. \\ 5x + 2y + 7z = 12. \end{cases}$

$$5.\quad \begin{cases} x+y+z=9. \\ x+y-z=5. \\ x-y+z=3. \\ -x+y+z=1. \end{cases}$$

62. If n simple simultaneous equations involving n unknowns are not independent, the value of each unknown takes the form $\frac{0}{0}$; i.e., is indeterminate.

For the determinant numerators and denominators will contain identical lines, or may be transformed into determinants containing identical lines.

Illustrative Solution.—

$$ax+by+cz=m.$$
$$bx+cy+az=n.$$
$$(a+b)\,x+(b+c)y+(a+c)z=m+n.$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ a+b & b+c & c+a \end{vmatrix} = 0.$$

$$x = \frac{\begin{vmatrix} m & b & c \\ n & c & a \\ m+n & b+c & c+a \end{vmatrix}}{0} = \frac{0}{0}:$$

$$y = \frac{\begin{vmatrix} a & m & c \\ b & n & a \\ a+b & m+n & c+a \end{vmatrix}}{0} = \frac{0}{0}.$$

This shows that any one of the unknowns in the given set of equations may have an indefinite number of values. If we assign a fixed value to one of them, the others will also become fixed. The values of the unknowns are just as indeterminate as if only two equations were given.

Homogeneous Equations of the First Degree.

63. *Definition.* — A homogeneous equation of the first degree is an equation all of whose terms contain an unknown factor of the first degree; e.g., $2x + 3y - z = 0$ and $x - y = 0$. $2x + 3y - z = 5$ is not homogeneous.

64. Theorem When the number of homogeneous equations is equal to the number of unknowns, all the unknowns equal zero, unless one equation is dependent.

Demonstration. — Given

$$\begin{cases} ax + by = 0. \\ cx + dy = 0. \end{cases}$$

Then $x = \frac{by}{a} = \frac{dy}{c}$; $\therefore$ if x and y are not equal to zero, $\frac{b}{a} = \frac{d}{c}$. Since the ratio of the coefficients is the same in the two equations, one may be obtained by multiplying the other by the ratio between the coefficients of the same unknown; i.e., one equation is dependent. When there are more than two equations we may reduce them to two by elimination; $\therefore$ the proof is general.

Illustrative Solution. —

$$\begin{cases} 2x - 3y + z = 0. \\ x + y + 2z = 0. \\ x - y - z = 0. \end{cases}$$

Eliminating z, we obtain

$$3x - 4y = 0.$$
$$3x - y = 0.$$

which are satisfied only when $x = y = 0$.

65. Any group of simple equations is satisfied by making all the unknowns equal to $\pm$ infinity, since the ratios between infinities are indeterminate. Any group of simple homogeneous equations is satisfied by making all the unknowns equal to

0 or $\pm\infty$. In general work we are not concerned with the infinite values, but seek only the finite quantities, which satisfy the equation or set of equations. In homogeneous equations we generally seek for values differing from zero.

66. When the number of homogeneous equations is one less than the number of unknowns, or when the number of equations is equal to the number of unknowns, but one of the equations is dependent, etc., the values of the unknowns may generally be found in terms of one of the unknowns; or, what is the same, the ratio which exists between the unknowns may be determined.

Consider the three homogenous equations

$$\begin{cases} a_1x + b_1y + c_1z = 0. \\ a_2x + b_2y + c_2z = 0. \\ a_3x + b_3y + c_3z = 0. \end{cases}$$

If z is not 0, we obtain the following:

$$\begin{cases} a_1\dfrac{x}{z} + b_1\dfrac{y}{z} = -c_1. \\ a_2\dfrac{x}{z} + b_2\dfrac{y}{z} = -c_2. \\ a_3\dfrac{x}{z} + b_3\dfrac{y}{z} = -c_3. \end{cases}$$

Since we have three equations and only two unknowns, one of the equations is dependent if they are consistent (59). The condition that these equations are consistent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

From this we deduce the important principle.

67. *In order that n homogeneous equations, involving n unknowns, may be simultaneous*, it is necessary and sufficient that the determinant formed of the coefficients of the unknowns in order (the determinant of the system) equals 0.

Examples.

1. Which of these sets consists of simultaneous equations?

$$a. \quad \begin{cases} 2x + 2y - 3z = 0. \\ 8x - y - z = 0. \\ 34x - 2y - 7z = 0. \end{cases}$$

$$b. \quad \begin{cases} x + 2y - 3z = 0. \\ 2x - y + 2y = 0. \\ x - 3y - z = 0. \end{cases}$$

Factors of a Determinant.

68. When a determinant is equal to zero after x is substituted for y, $y - x$ is one of its factors.

Demonstration. — Let Δ and Δ' represent the determinant before and after substituting.

Since Δ has some terms which contain y^n as a factor by hypothesis, we have

$$\Delta = Sy^0 + S_1 y + S_2 y^2 + S_3 y^3 + \ldots$$

Where S, S_1, S_2, . . . are the coefficients of the different powers of y and independent of y, therefore they are unaltered by the substitution of x for y. $\therefore$

$$\Delta' = Sx^0 + S_1 x + S_2 x^2 + S_3 x^3 + \ldots$$

Subtracting, remembering that $\Delta' = 0$, we obtain

$$\Delta = S_1 (y - x) + S_2 (y^2 - x^2) + \\ S_3 (y^3 - y^3) + \ldots$$

Hence Δ is divisible by $y - x$. The elements of Δ are supposed to contain only positive integral powers of y.

Illustrative Solution. —

$$\text{Given, } \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ a^2 & a & 1 \end{vmatrix}$$

Explanation. — When we substitute y for x, two lines become identical. Therefore $x - y$ is a factor. For the same reason $x - a$, $y - a$ are factors. It is not difficult to see that these are all the factors, and that the determinant is equal to $(x - y)(y - a)(x - a)$.

The given determinant could have been factored by subtracting the rows separately; e.g., subtracting the second row from the first, we find that $x - y$ is a factor; subtracting the third row from the first, and we see that $x - a$ is a factor; by subtracting the third row from the second, we find the remaining factor $y - a$. These successive transformations, which leave the determinant unaltered, are often very useful to discover the factors. We shall give one more example where the principle may be employed with advantage.

Prove that $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} =$

$$(a+b+c+d)\,(a+b-c-d)\times$$
$$(a-b+c-d)\,(a-b-c+d).$$

EXERCISE.

1. Eliminate the unknowns from

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0. \\ a_2x + b_2y + c_2z + d_2 = 0. \\ a_3x + b_3y + c_3z + d_3 = 0. \\ a_4x + b_4y + c_4z + d_4 = 0. \end{cases}$$

Suggestion. — Find the values of the unknowns from the first three equations and substitute these values for the unknowns in the fourth equation. or put the Δ formed of coefficients & absolute terms $= 0$

2. Prove that

$$\begin{vmatrix} \omega^2 & 1 & \omega & 1 \\ 1 & \omega & \omega^2 & 1 \\ 1 & \omega^2 & 1 & \omega \\ \omega & 1 & 1 & \omega^2 \end{vmatrix} \pm 3\sqrt{-3}.$$

ω being one of the imaginary cube roots of unity.

3. If all the elements on one side of the principal diagonal are 0's, what is the value of the determinant?

4. If all of the elements on one side of the secondary diagonal are 0's, what is the value of the determinant?

5. Employing the notation

$$\binom{n}{l} = \frac{n(n-1)(n-2)\ldots(n-l+1)}{l!}.$$

Prove that

$$\begin{vmatrix} 1 & \binom{a+b}{1} & \binom{a+b+1}{2} & \binom{a+b+2}{3} & \cdots & \binom{a+2b-1}{b} \\ 1 & \binom{a+b+1}{1} & \binom{a+b+2}{2} & \binom{a+b+3}{3} & \cdots & \binom{a+2b}{b} \\ 1 & \binom{a+b+2}{1} & \binom{a+b+3}{2} & \binom{a+b+4}{3} & \cdots & \binom{a+2b+1}{b} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{a+2b}{1} & \binom{a+2b+1}{2} & \binom{a+2b+2}{3} & \cdots & \binom{a+3b-1}{b} \end{vmatrix} = 1.$$

Suggestion. — Subtracting the $(n-1)^{th}$ row from the n^{th}, the $(n-2)^{th}$ from the $(n-1)^{th}$, etc., we obtain

$$\begin{vmatrix} 1 & \binom{a+b}{1} & \binom{a+b+1}{2} & \binom{a+b+2}{3} & \cdots & \binom{a+2b-1}{b} \\ 0 & 1 & \binom{a+b+1}{1} & \binom{a+b+2}{2} & \cdots & \binom{a+2b-1}{b-1} \\ 0 & 1 & \binom{a+b+2}{1} & \binom{a+b+3}{2} & \cdots & \binom{a+2b}{b-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \binom{a+2b}{1} & \binom{a+2b+1}{2} & \cdots & \binom{a+3b-2}{b-1} \end{vmatrix}.$$

Continue $b-1$ times. Observe that

$$\binom{n}{r} - \binom{n-1}{r} = \binom{n-1}{r-1}.$$

Multiplication of Determinants.

69. The notation $\begin{vmatrix} a & b' \\ 0 & b \end{vmatrix}$ indicates the product of a and b.

When a and b are themselves determinants we see one rule for determinant multiplication; as follows: —

Place the determinants so that their principal diagonals together form the principal diagonal of a new determinant. Fill the places in one of the empty squares with 0's, and in the other with *any* elements. The resulting determinant is the product of the given determinants.

Illustration. —

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 5 & 1 \\ 0 & 2 & 1 \end{vmatrix} \times \begin{vmatrix} 7 & -4 \\ 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & 4 & 0 & -1 \\ 1 & 5 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -6 \\ 0 & 0 & 0 & 7 & -4 \\ 0 & 0 & 0 & 3 & 4 \end{vmatrix} = 440.$$

Solution. —

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 5 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} -4 & 5 \\ -2 & -3 \end{vmatrix} = 11.$$

$$\begin{vmatrix} 7 & -4 \\ 3 & 4 \end{vmatrix} = 40. \qquad 11 \times 40 = 440.$$

$$\begin{vmatrix} 2 & 3 & 4 & 0 & 1 \\ 1 & 5 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 6 \\ 0 & 0 & 0 & 7 & -4 \\ 0 & 0 & 0 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -11 & 0 & 0 & -11 \\ 1 & 5 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 6 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 3 & 4 \end{vmatrix}$$

$$= -1 \times 10 \times 1 \begin{vmatrix} -11 & -11 \\ 0 & 4 \end{vmatrix} = 440.$$

Th - . m

70. The product of two determinants of the n^{th} order is a determinant of the n^{th} order.

We shall multiply two determinants of the third order. The learner can easily see that the method employed is general.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} d_1 & e_1 & l_1 \\ d_2 & e_2 & l_2 \\ d_3 & e_3 & l_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & d_1 & e_1 & l_1 \\ 0 & -1 & 0 & d_2 & e_2 & l_2 \\ 0 & 0 & -1 & d_3 & e_3 & l_3 \end{vmatrix}$$

$$
= \left| \begin{array}{cccc}
a_1 & b_1 & c_1 & a_1d_1 + b_1d_2 + c_1d_3 \\
a_2 & b_2 & c_2 & a_2d_1 + b_2d_2 + c_2d_3 \\
a_3 & b_3 & c_3 & a_3d_1 + b_3d_2 + c_3d_3 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{array} \right.
$$

$$
\left. \begin{array}{cc}
a_1e_1 + b_1e_2 + c_1e_3 & a_1l_1 + b_1l_2 + c_1l_3 \\
a_2e_1 + b_2e_2 + c_2e_3 & a_2l_1 + b_2l_2 + c_2l_3 \\
a_3e_1 + b_3e_2 + c_3e_3 & a_3l_1 + b_3l_2 + c_3l_3 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{array} \right| .
$$

By 42 this reduces to

$$
\left| \begin{array}{ll}
a_1d_1 + b_1d_2 + c_1d_3 & a_1e_1 + b_1e_2 + c_1e_3 \\
a_2d_1 + b_2d_2 + c_2d_3 & a_2e_2 + b_2e_3 + c_2e_3 \\
a_3d_1 \quad\; b_3d_2 + c_3d_3 & a_3e_1 + b_3e_3 + c_3e_3
\end{array} \right.
$$

$$
\left. \begin{array}{c}
a_1l_1 + b_1l_2 + c_1l_3 \\
a_2l_1 + b_2l_2 + c_2l_2 \\
a_3l_1 + b_3l_2 + c_3l_3
\end{array} \right| .
$$

Making the rows of $(d_1\ e_2\ l_3)$ its columns (34), we find the following rule for the multiplication of determinants of the same order.

Determinant Multiplication.

Method. — Multiply the elements of the first row of Δ [1] by the corresponding elements of the first row of Δ', the sum of the products is the first element in the first column of Δ''.

Multiply the elements of the second row of Δ by the corresponding elements of the first row of Δ'. The sum of the products is the second element in the first column of Δ''.

Multiply the elements of the remaining rows of Δ by the corresponding elements of the first row of Δ'. The sums of the products are the third, fourth, etc., elements in the first column of Δ''.

To find the elements of the second column of Δ'', we multiply by the elements of the second row of Δ', and proceed in a similar way, etc.

The method may be easily learned by studying the following solutions, and then performing the operations without referring to the book.

[1] $\Delta =$ multiplicand; $\Delta' =$ multiplier; $\Delta'' =$ product.

1. $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \times \begin{vmatrix} 1 & -2 \\ 9 & 7 \end{vmatrix} =$

$$\begin{vmatrix} 2.1 - 3.2 & 2.9 + 3.7 \\ 1.1 - 4.2 & 1.9 + 4.7 \end{vmatrix} = 125.$$

2. $\begin{vmatrix} 2 & 4 & 3 \\ 1 & 0 & 2 \\ 1 & 4 & 2 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix} =$

$$\begin{vmatrix} 2.1 + 4.2 + 3.4 & 2.2 + 4.3 + 3.1 \\ 1.1 + 0.2 + 2.4 & 1.2 + 0.3 + 2.1 \\ 1.1 + 4.2 + 2.4 & 1.2 + 4.3 + 2.1 \end{vmatrix}$$

$$\begin{vmatrix} 2.1 + 4.0 + 3.2 \\ 1.1 + 0.0 + 2.2 \\ 1.1 + 4.0 + 2.2 \end{vmatrix} = \begin{vmatrix} 22 & 19 & 8 \\ 9 & 4 & 5 \\ 17 & 16 & 5 \end{vmatrix}.$$

71. By transforming Δ and Δ' according to 34, Δ'' will assume eight different forms.

Demonstration. — The product may be obtained by any one of the following methods. $r \times r'$, $r \times c'$, $c \times r'$, $c \times c'$, $r' \times r$, $r' \times c$, $c' \times r$, $c' \times c$: where $r, c =$ rows and columns of Δ, and $r', c' =$ rows and columns of Δ'.

Illustrative Solution.—

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \times \begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae + bf & ag + bh \\ ce + df & cg + dh \end{vmatrix} =$$

$$\begin{vmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{vmatrix} = \begin{vmatrix} ae + cf & ag + ch \\ be + df & bg + dh \end{vmatrix}$$

$$= \begin{vmatrix} ae + cg & af + ch \\ be + dg & bf + dh \end{vmatrix}$$

$$= \begin{vmatrix} ae + bf & ce + df \\ ag + bh & cg + dh \end{vmatrix}$$

$$= \begin{vmatrix} ae + bg & ce + dg \\ af + bh & cf + dh \end{vmatrix}$$

$$= \begin{vmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{vmatrix}$$

$$= \begin{vmatrix} ae + cg & be + dg \\ af + ch & bf + dh \end{vmatrix}.$$

Observations.—All of these products may be obtained by multiplying rows by rows, according to method, page 98, after Δ and Δ' have been transformed (34). All of the products equal $aedh + bcfg - adfg - bceh$. Interchanging multiplier and multi-

plicand interchanges the rows and columns of the product. The product will assume the same number of forms when the determinants are of a higher order.

72. When the determinants to be multiplied are of different orders, they should be transformed into determinants of the same order before multiplying (43, 45).

Exercise.

Prove that

1. $$\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \times \begin{vmatrix} 5 & 6 \\ 8 & 7 \end{vmatrix} = \begin{vmatrix} 28 & 37 \\ 26 & ? \end{vmatrix} = 130.$$

2. $$\begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 4 & 1 & 8 \\ 4 & 1 & 2 & 13 \\ 2 & 4 & 2 & 11 \end{vmatrix} \times \begin{vmatrix} 2 & 4 & -3 \\ 4 & -2 & 5 \\ 6 & 7 & -1 \end{vmatrix} = 750.$$

3. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} A_1 & -B_1 & C_1 \\ -A_2 & B_2 & -C_2 \\ A_3 & -B_3 & C^3 \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

Note. — It is worthy of observation that a determinant is not altered by changing the signs of all the elements whose row numbers ~~and~~ increased by its column numbers ~~are~~ odd. If the order of the determinant is even the signs of all the elements may be changed.

Suggestion. —

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & -B_1 & C_1 \\ -A_2 & B_2 & -C_2 \\ A_3 & -B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^3 \quad (52, 53).$$

4. $$\begin{vmatrix} 2 & 3 & 1 \\ 1 & 4 & 5 \\ 5 & 12 & 14 \end{vmatrix}^2$$

$$= \begin{vmatrix} \begin{vmatrix} 4 & 5 \\ 12 & 14 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 5 & 14 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 5 & 12 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 12 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 5 & 14 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \end{vmatrix}.$$

5. $$\begin{vmatrix} A_1 & B_1 & \dots & L_1 \\ A_2 & B_2 & \dots & L_2 \\ \dots & \dots & \dots & \dots \\ A_n & B_n & \dots & L_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & \dots & c_1 \\ a_2 & b_2 & \dots & c_2 \\ \dots & \dots & \dots & \dots \\ a_n & b_n & \dots & c_n \end{vmatrix}^{n-1}$$

Suggestion. — The product of $(a_1\ b_2 \ldots c_n) \times (A_1\ B_2 \ldots L_n)$ is $(a_1\ b_2 \ldots c_n)^n$; dividing by $(a_1\ b_2 \ldots c_n)$, we obtain the required result.

6. $$\begin{vmatrix} a - ib & -c + id \\ c + id & a + ib \end{vmatrix} \times \begin{vmatrix} e - if & -g + ih \\ g + ih & e + if \end{vmatrix}$$

$$= \begin{vmatrix} A - iB & -C + iD \\ C + iD & A + iB \end{vmatrix}.$$

$$i \equiv \sqrt{-1}.$$
$$A \equiv ae - bf + gc - dh.$$
$$B \equiv af + be + ch + dg.$$
$$C \equiv ce + df - ag - bh.$$
$$D \equiv ah + de - by - cf.$$

The result of problem 6 may be stated. The product of two sums, each consisting of four squares, is itself the sum of four squares. This is Euler's Theorem.

SPECIAL FORMS.

[1] 73. The general equation of the second degree, involving two variables, is frequently written as follows:

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

The condition that this equation represents two planes, or two straight lines in the xy plane, is that the following equations should be satisfied:

$$u = o, \frac{du}{dx} = 0, \frac{du}{dy} = 0.$$

In these equations u stands for the first member of the original equation. This condition may also be expressed as follows:

$$2u - x\frac{du}{dx} - y\frac{du}{dy} = 0, \frac{du}{dx} = 0, \frac{du}{dy} = 0.$$

From the theory of homogeneous functions (quantics), we know that the operation $2u - x\frac{du}{dx} - y\frac{du}{dy}$ will remove the terms of the second degree from u, hence

[1] Students not familiar with Differential Calculus may omit this article.

we can write the above conditions in the following form :

$$\begin{aligned} ax + hy + g &= 0, \\ hx + by + f &= 0, \\ gx + fy + c &= 0. \end{aligned}$$

The condition that these equations are consistent is (59) :

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

74. Using two subscripts with each element, the first to indicate the row and the second the column in which the element occurs, Δ may be written thus,

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1\,n} \\ a_{21} & a_{22} & \dots & a_{2\,n} \\ \dots & \dots & \dots & \dots \\ a_{n\,1} & a_{n\,2} & \dots & a_{nn} \end{vmatrix}.$$

a_{11}, a_{22}, a_{33}, . . . a_{nn} and $a_{n\,1}$, $a_{n-1\,2}$, $a_{n-2\,3}$, . . . $a_{1\,n}$ are the elements in the Principal and Secondary Diagonals respectively.

Employing this notation for the last determinant in the preceding article, we have $a_{12} = a_{21}$, $a_{13} = a_{31}$, and $a_{23} = a_{32}$. If in

a determinant $a_{ik} = a_{ki}$, where $_k$ and $_i$ have any positive integral values from ~~$_1$~~ n to ~~$_n$~~ n the determinant is said to be a *symmetrical determinant.* The last Δ in the preceding article is a symmetrical Δ.

75. Symmetrical determinants occur very frequently. We proceed to study a method by which they may be readily developed. We shall first develop the general Δ.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2\,n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n\,1} & a_{n\,2} & a_{n\,3} & \dots & a_{nn} \end{vmatrix}.$$

correct

Designating the co-factor~~s~~ (50) of the elements of the principal minor of Δ corresponding to a_{11} by β_{ik}; e.g., $\beta_{34} = -(a_{22}\, a_{43}\, a_{55} \dots a_{nn})$, it is seen directly that,

$$\Delta = a_{11}\,\mathrm{A}_{11} - \{a_{21}\,a_{12}\,\beta_{22} + a_{21}\,a_{13}\,\beta_{23} + a_{21}\,a_{14}\,\beta_{24} + \dots + a_{21}\,a_{1\,n}\,\beta_{2\,n} + a_{31}\,a_{12}\,\beta_{32} + a_{31}\,a_{13}\,\beta_{33} + a_{31}\,a_{14}\,\beta_{34} + \dots + a_{31}\,a_{1\,n}\,\beta_{3\,n} + \dots\}.$$

This expansion is called Cauchy's Theorem. It may be written in the form,

$$\Delta = a_{11}\, A_{11} - \Sigma\, a_{i\,1}\, a_{1\,k}\, \beta_{ik}$$

i and $k = 2, 3, 4, \ldots n$.

When Δ is symmetrical, this formula becomes,

$$\Delta = a_{11}\, A_{11} - \Sigma\, a_{i1}^2\, \beta_{ii} - 2\, \Sigma\, a_{i\,1}\, a_{k\,1}\, \beta_{ik}.$$

In the last determinant in (72) $A_{11} = bc - f^2$; $\beta_{ii} = c, b$; $\beta_{ik} = -f$. Hence its development is $abc - af^2 - h^2c - g^2b + 2fgh$.

Elimination.

76. Given,

$$ax^m + bx^{m-1} + cx^{m-2} + \ldots + l = 0,$$
$$a^1\, x^n + b^1\, x^{n-1} + c^1\, x^{n-2} + \ldots + l^1 = 0.$$

We may eliminate x and find the relation between the constant coefficients as follows:

Multiply the first equation by x, $n - 1$ times; and the second $m - 1$ times; and eliminate x from the $m + n$ resulting equations; e.g.:

$$ax^3 + bx^2 + cx + d = 0.$$
$$px^2 + qx + r = 0.$$

The $m + n$ equations are,

$$\left.\begin{aligned} px^2 + qx + r &= 0, \\ px^3 + qx^2 + rx &= 0, \\ px^4 + qx^3 + rx^2 &= 0, \\ ax^3 + bx^2 + cx + d &= 0, \\ ax^4 + bx^3 + cx^2 + dx &= 0. \end{aligned}\right\}$$

The condition of consistency is,

$$\begin{vmatrix} 0 & 0 & p & q & r \\ 0 & p & q & r & 0 \\ p & q & r & 0 & 0 \\ 0 & a & b & c & d \\ a & b & c & d & 0 \end{vmatrix} = 0.$$

This method is known as Sylvester's Dialytic Method. It is not necessary to regard the coefficients of x constants. They may be functions of variables.

Direction. — Eliminate x from the following equations :

1. $$\left.\begin{aligned} (y-1)\,x^2 + yx + y &= 0, \\ yx - y &= 2. \end{aligned}\right\}$$

2. $$\left.\begin{aligned} 3\,x^3 - yx + 4\,y &= 0, \\ xy &= 12. \end{aligned}\right\}$$

3. $$\left.\begin{aligned} yx^3 - (y^3 - 3y - 1)\,x + y &= 0, \\ x^2 - y^2 + 3 &= 0. \end{aligned}\right\}$$

4. $$\left.\begin{aligned} x^2 - y &= 7 + x^3 - 3\,xy, \\ x + y^2 &= 10 - 2\,xy. \end{aligned}\right\}$$

Note A.

If the determinant of the second order vanishes, i.e., if

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

we must have $a_1 b_2 - a_2 b_1 = 0$. Dividing this by $a_2 b_2$, we obtain, in general,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}.$$

This equation indicates that the first row is dependent upon the second.

If the determinant of the third order vanishes, i.e., if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

we may add to the first row multiples of the other two rows, so as to make at least two elements in the first row vanish, since the equations

$$a_2 x + a_3 y - a_1 = 0,$$
$$b_2 x + b_3 y - b_1 = 0,$$

can be solved. If the values of x and y thus found be represented by x_1 and y_1, and the equation

$$c_2 x_1 + c_3 y_1 - c_1 = 0$$

is satisfied, the first row of Δ is dependent upon the other two rows. When the last equation is

not satisfied, we obtain, by developing Δ' in terms of the principal minors of the first row,

$$(c_2x_1 + c_3y - c_1)\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

Since the first factor is by hypothesis, finite $C_1 = 0$, it can be shown in a similar manner that $A_1 = B_1 = 0$ at the same time. Hence, if $\Delta = 0$, either the first row must be dependent upon the other two, or the principal minors corresponding to the elements of the first row must all vanish; i. e., $A_1 = B_1 = C_1 = 0$.

When the last condition is fulfilled, the last two rows can be shown to be dependent by the method employed in the first part of this note; and hence, in either of the two cases, when $\Delta = 0$, one row, at least, is dependent. When all the principal minors corresponding to the elements of two parallel lines in Δ vanish, two rows of Δ must be dependent. This method of proof may readily be applied to determinants of higher orders.

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$\Pi(r-i)(r-k)$ remains unchanged if i & k are interchanged. For only the intermediate elements ~~letters~~ will lead to factors whose sigs are changed. Assuming that $a+b = a'+b' = n =$ the number of elements between i and k and that a & a' represent the numbers of those larger than k and i respectively, while b and b' represent the numbers of those that are smaller than k & i respectively, assuming also that i precedes k, we

have in the original arrangement $a+f'$ negative factors and in the new arrangement $a'+b$ negative factors. It remains to show that the difference of these sums is even. That is that $a-a'+f'-f$ is even. From the ~~original~~ given assumed equation

$$a+b=a'+b'$$

we obtain

$$a-a'=b'-b$$

Hence the above number is equal to $2(a-a')$ & therefore even.

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Alternative proof —
Consider the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

And it be required to reduce all the elements coeff. in first in the first row to 0's, we have the following equations

$$a_4 x + a_3 y + a_2 z + a_1 w = 0$$
$$b_4 x + b_3 y + b_2 z + b_1 w = 0$$
$$c_4 x + c_3 y + c_2 z + c_1 w = 0$$
$$d_4 x + d_3 y + d_2 z + d_1 w = 0$$

Since we have four equations & four unknowns the eqs can be solved and an obvious ex